The partial McKay correspondence

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Summary

McKay Correspondence $\longrightarrow$ Tilting bundle $\mathcal{V}$ on S $\longrightarrow$ Equivalence of categories

$$D^b(\mathrm{coh}\, S) \to D^b(\mathrm{End}\,\mathcal{V}\text{-mod})$$

Summary

Partial McKay Correspondence $\longrightarrow$ Tilting bundle $\cancel{\mathcal{X}}_{T}$ on $\cancel{\mathcal{S}}_{S_K}$ $\longrightarrow$ Equivalence of categories

$$D^b(\mathrm{coh}\,\cancel{\mathcal{S}}_{S_K}) \to D^b(\mathrm{End}\,\cancel{\mathcal{X}}_{T}\text{-mod})$$

ADE singularities & partial resolutions

- Quotients of $\mathbb{C}^2$ by $\Gamma \subset SL(2, \mathbb{C})$ finite
- $\mathbb{C}^2/\Gamma := \operatorname{Spec}\left(\mathbb{C}[x, y]^{\Gamma}\right)$

ADE singularities & partial resolutions

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ADE singularities & partial resolutions

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- $\mathbb{C}^2/\Gamma := \mathrm{Spec}\left(\mathbb{C}[x,y]^\Gamma\right) = \mathrm{Spec}\left(\mathbb{C}[u,v,w]/f\right)$
- 2 infinite families : A_r, D_r

  A_r → Correspond to $\Gamma \subset SL(2,\mathbb{C})$ cyclic.
- 3 exceptional cases : E_6, E_7, E_8

ADE singularities & partial resolutions

- Quotients of $\mathbb{C}^2$ by $\Gamma \subset SL(2,\mathbb{C})$ finite
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- 3 exceptional cases: E_6, E_7, E_8

Eg: A_2 sing:

$\mathbb{C}[x,y]^\Gamma = \mathbb{C}[u,v,w]/(uv-w^3)$

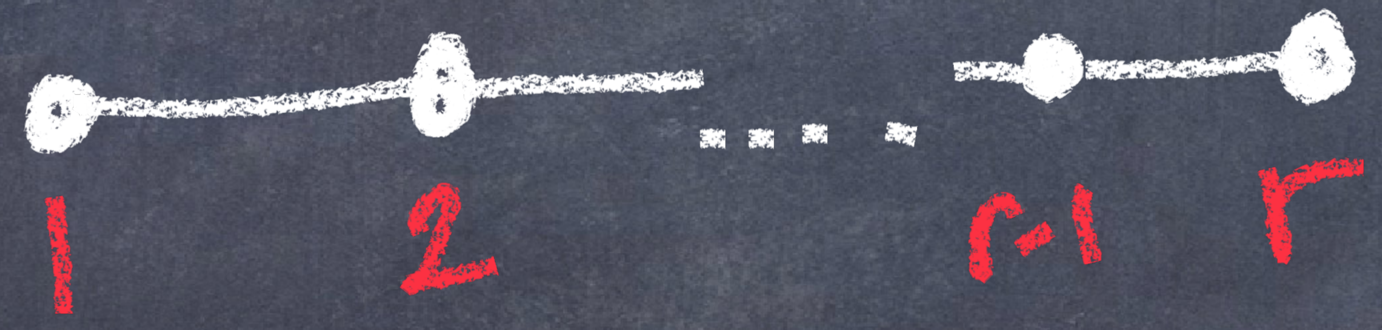
ADE singularities & partial resolutions

- $\mathbb{C}^2/\Gamma$ has an isolated singularity at the origin.
- Blow up to form minimal resolution $S \to \mathbb{C}^2/\Gamma$.
- Exceptional divisor has r $\mathbb{P}^1$ curves

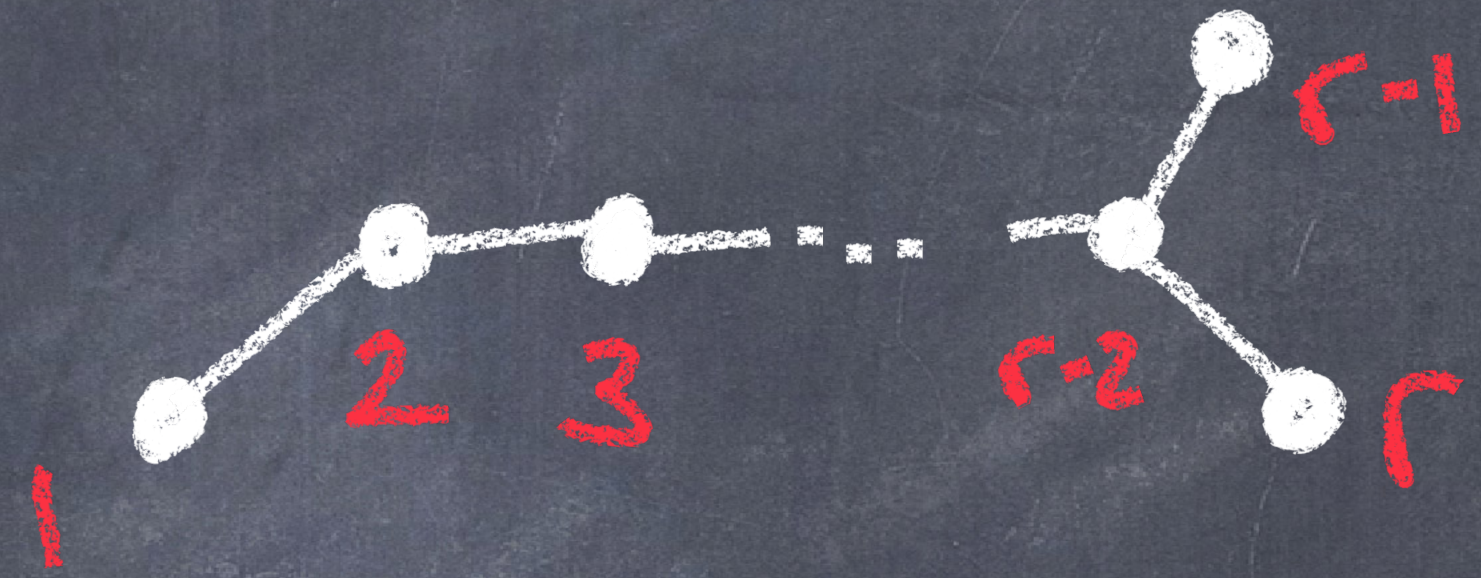
ADE singularities & partial resolutions

- Exceptional divisor has r $\mathbb{P}^1$ curves arranged:

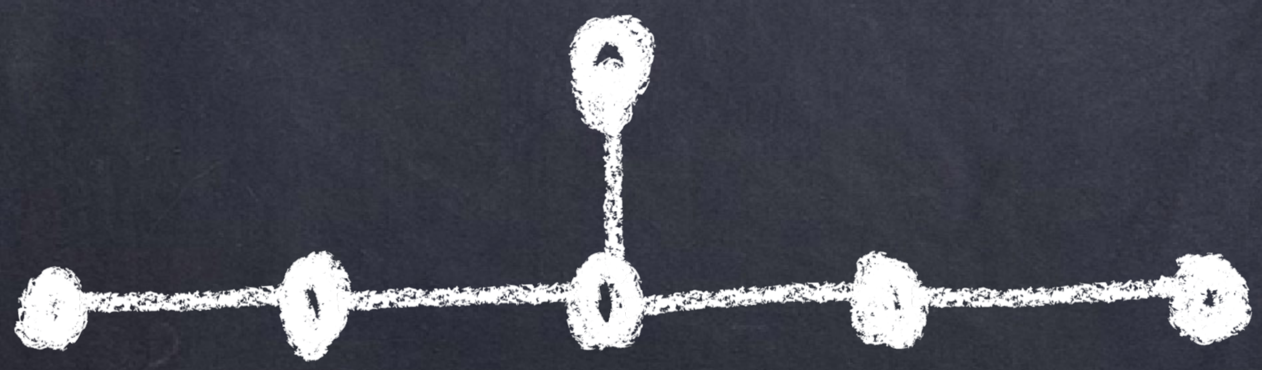
A_r :



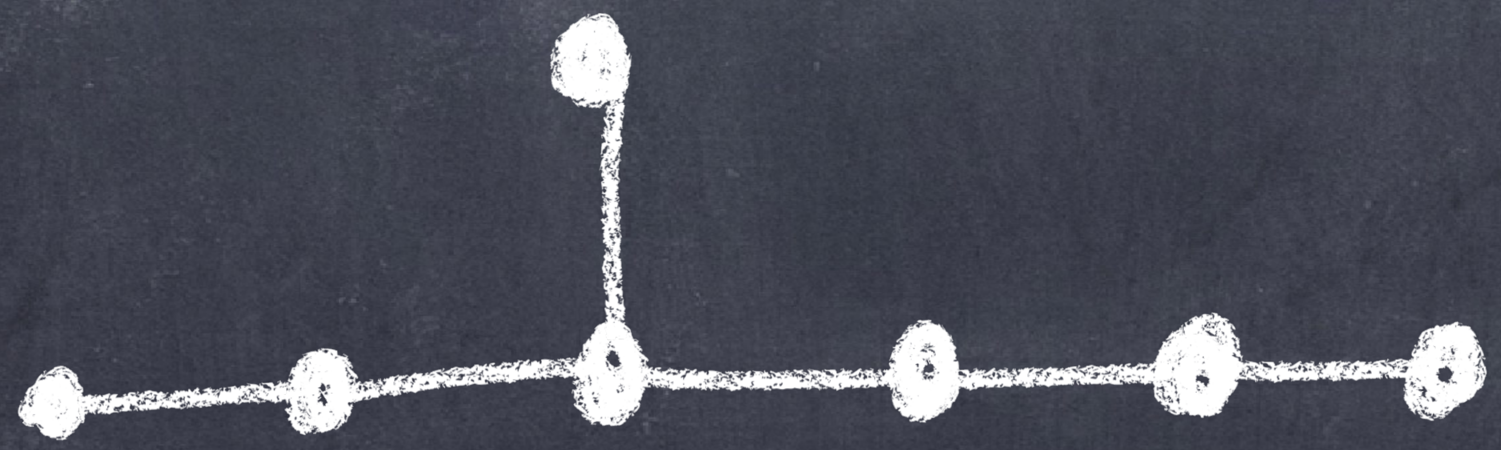
D_r :



E_6:



E_7:



E_8:

ADE singularities & partial resolutions

$$\begin{array}{c} S \quad (r\ \mathbb{P}^1\text{ curves}) \\ \big\downarrow \\ \mathbb{C}^2/\Gamma \end{array}$$

ADE singularities & partial resolutions

$$
\begin{tikzcd}
 & & S \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \text{(r $\mathbb{P}^1$ curves)} \\
\text{($r-1$ $\mathbb{P}^1$ curves)}\ S_1 \arrow[dr] & S_2 \arrow[d] & S_3 \arrow[dl] & \cdots \\
 & \mathbb{C}^2/\Gamma & &
\end{tikzcd}
$$

ADE singularities & partial resolutions

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\begin{tikzcd}
 & & & S \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \text{(r $\mathbb{P}^1$ curves)} \\
 & & & & \cdots \\
\text{(r-1 $\mathbb{P}^1$ curves)} & S_1 \arrow[d] & S_2 \arrow[dl] \arrow[d] & S_3 \arrow[dl] & \cdots \\
\text{(r-2 $\mathbb{P}^1$ curves)} & S_{12} \arrow[dr] & S_{23} \arrow[d] & \cdots & \\
 & & \mathbb{C}^2/\Gamma & &
\end{tikzcd}
$$

ADE singularities & partial resolutions

$$\begin{tikzcd}
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 & & \cdots & & \\
 & & \mathbb{C}^2/\Gamma & &
\end{tikzcd}$$

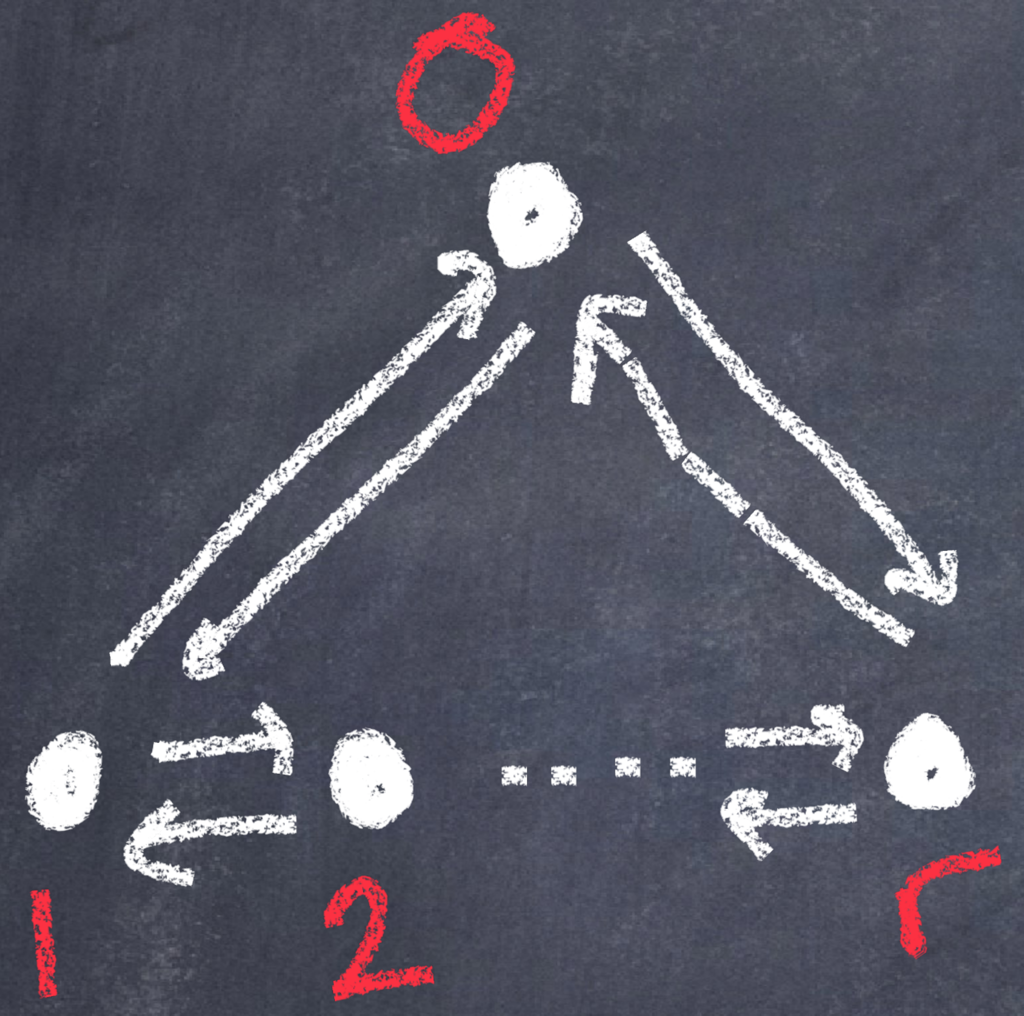
ADE singularities & partial resolutions

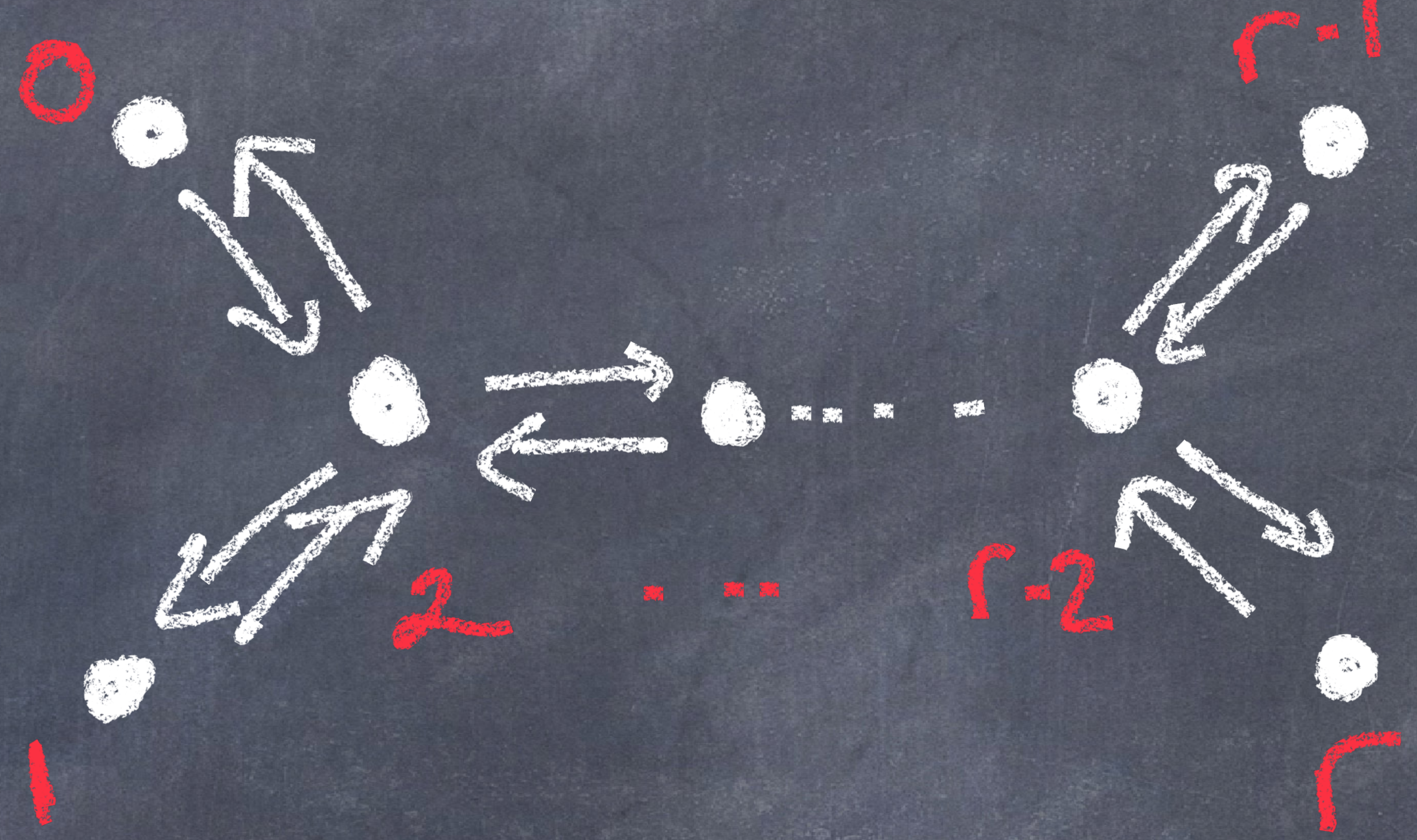
For $K \subseteq \{1, \ldots, r\}$

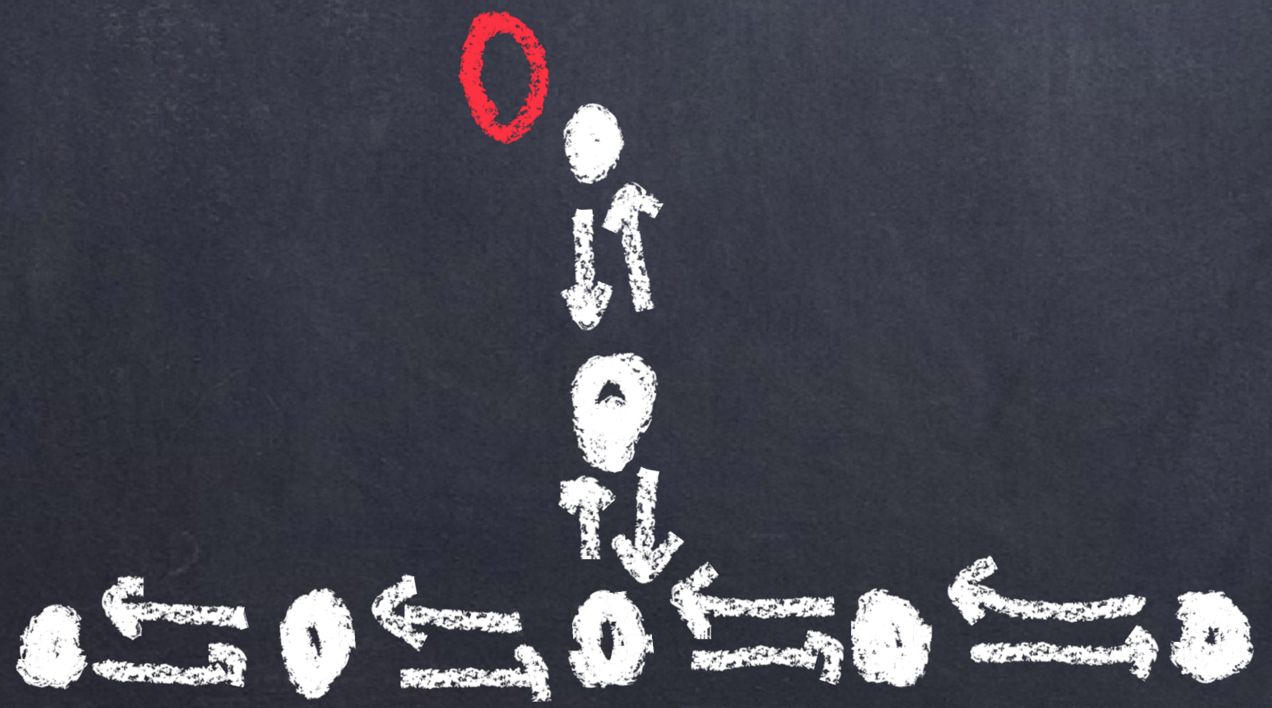
$$\begin{array}{c} S \\ \downarrow \\ S_K \\ \downarrow \\ \mathbb{C}^2/\Gamma \end{array}$$

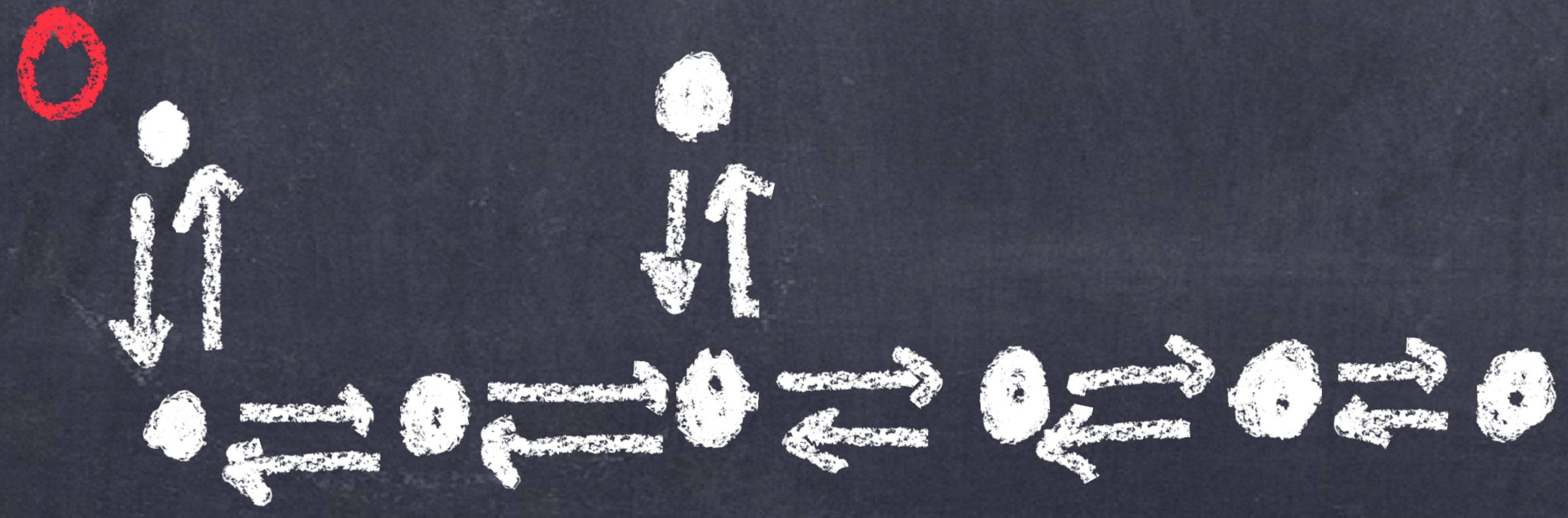
Quiver varieties

Quiver varieties

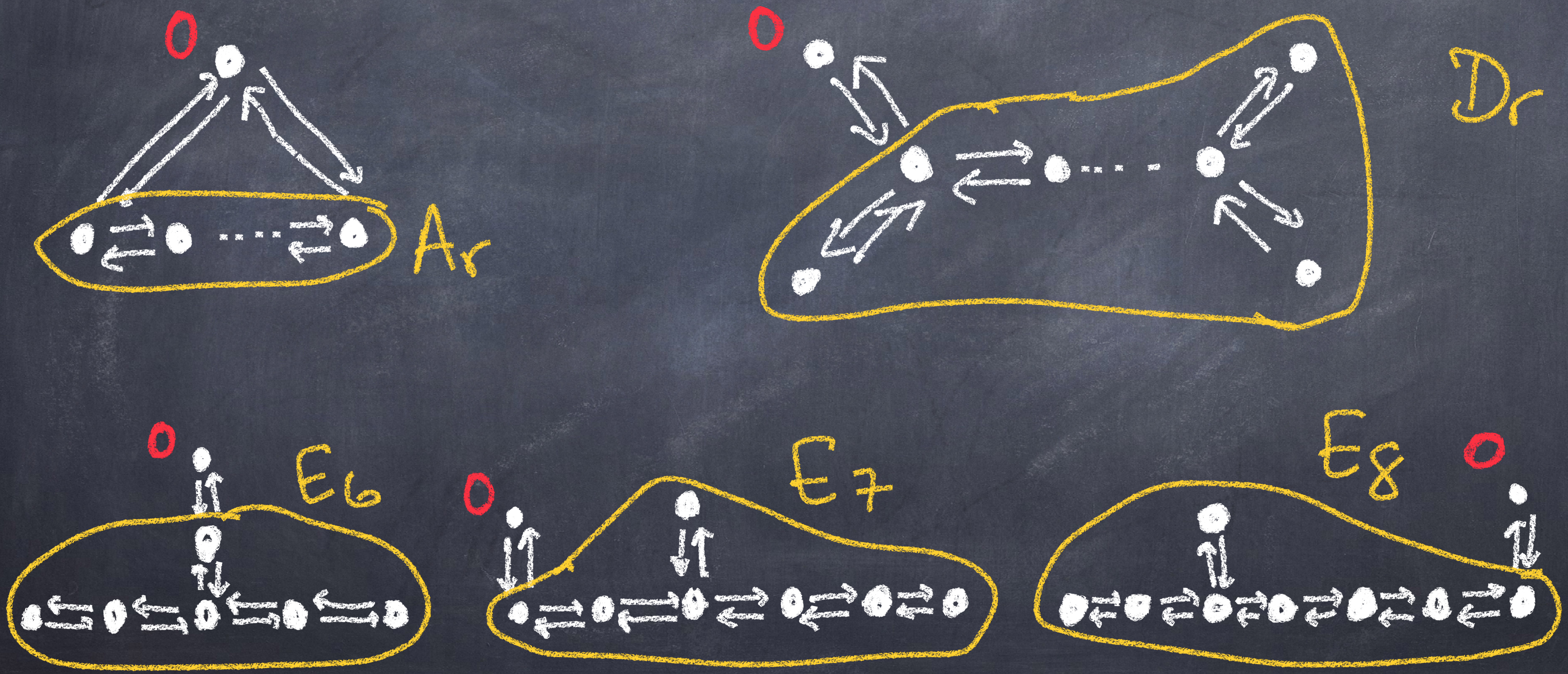








Quiver varieties



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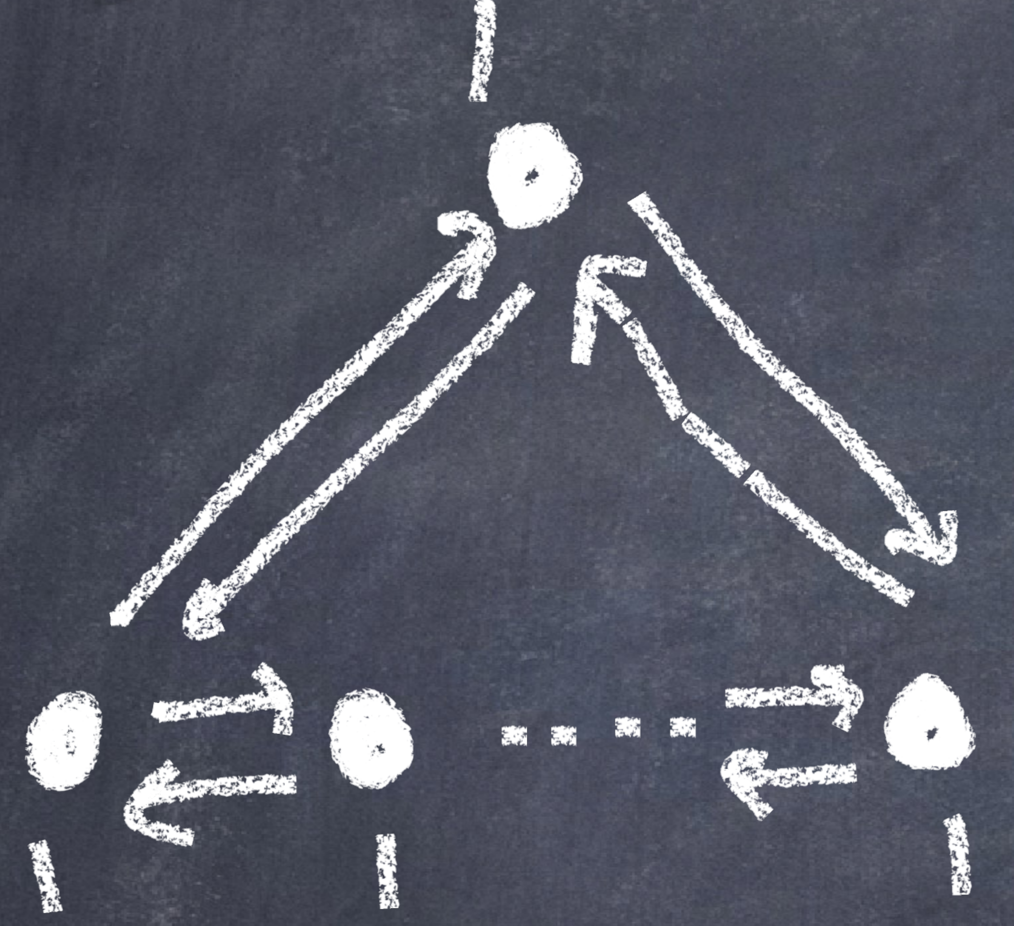

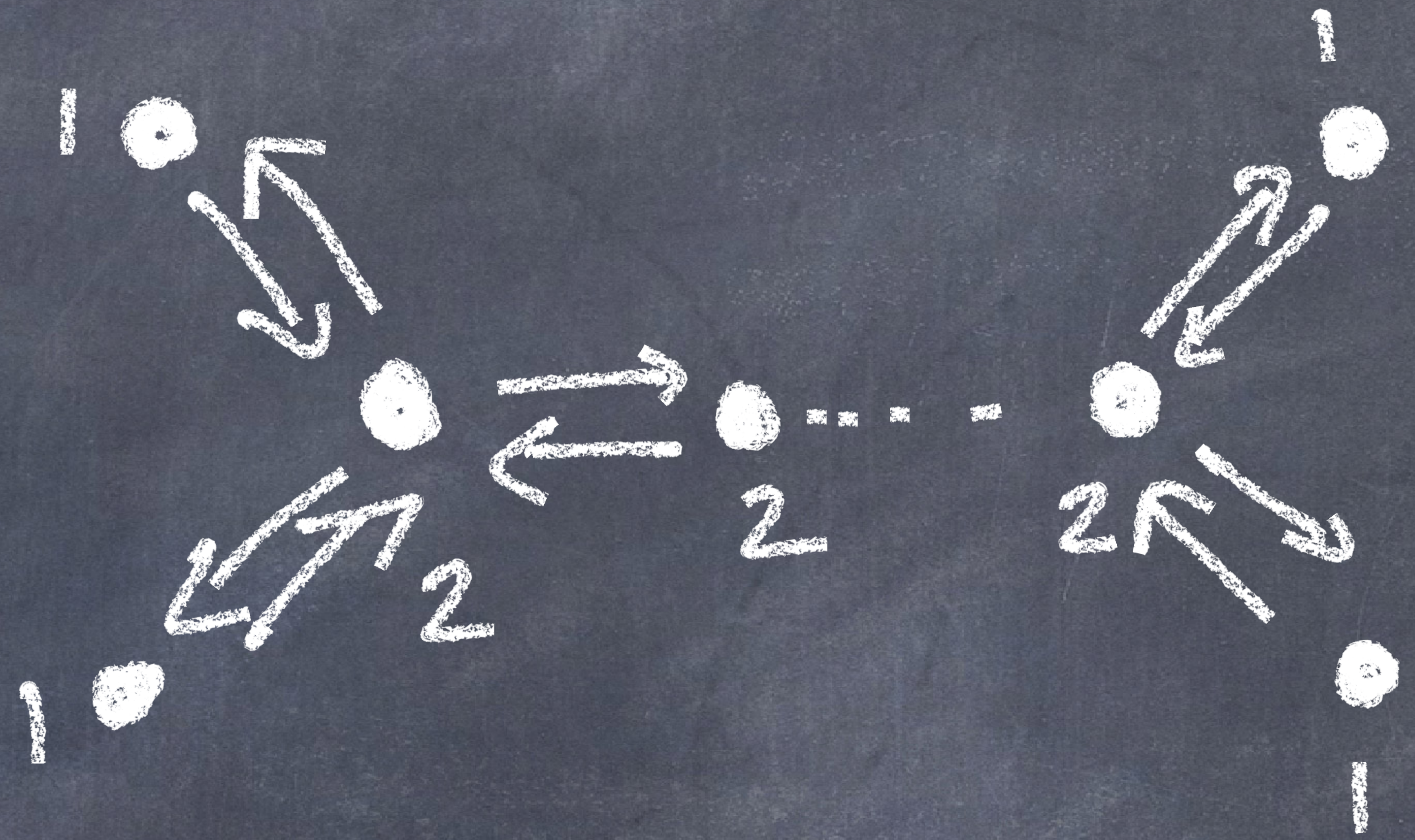

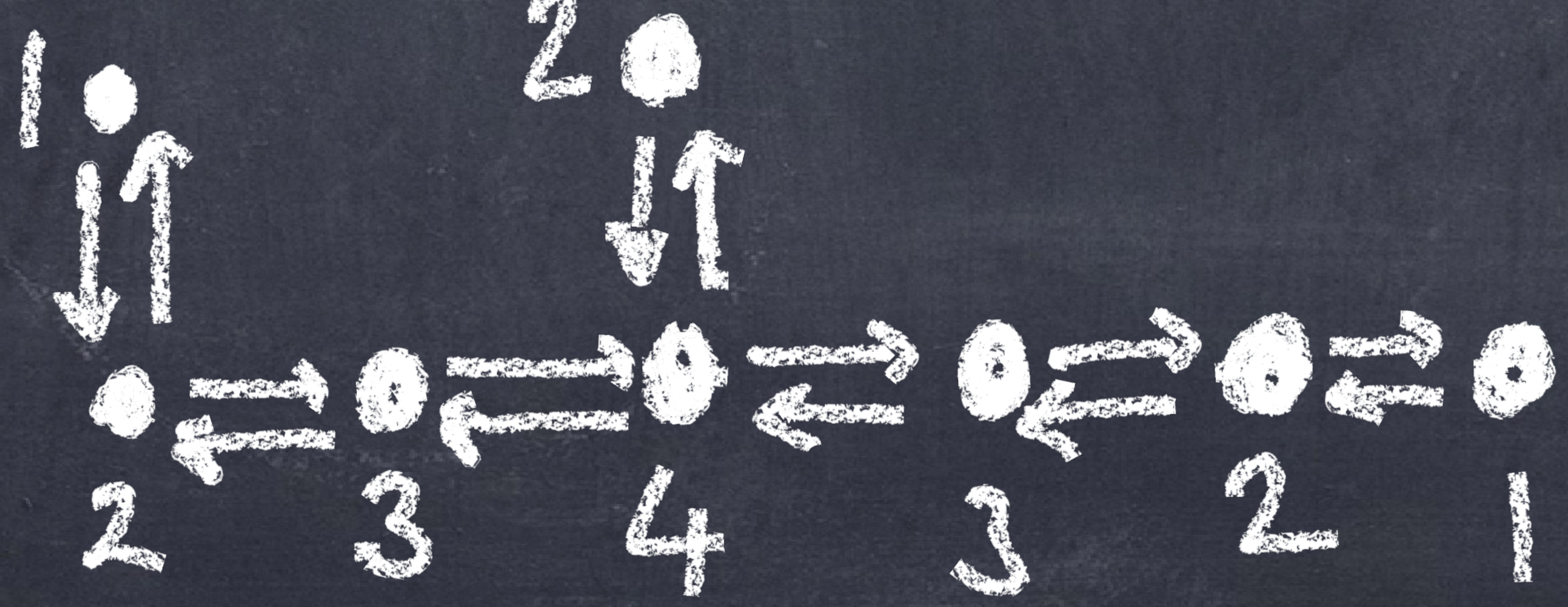

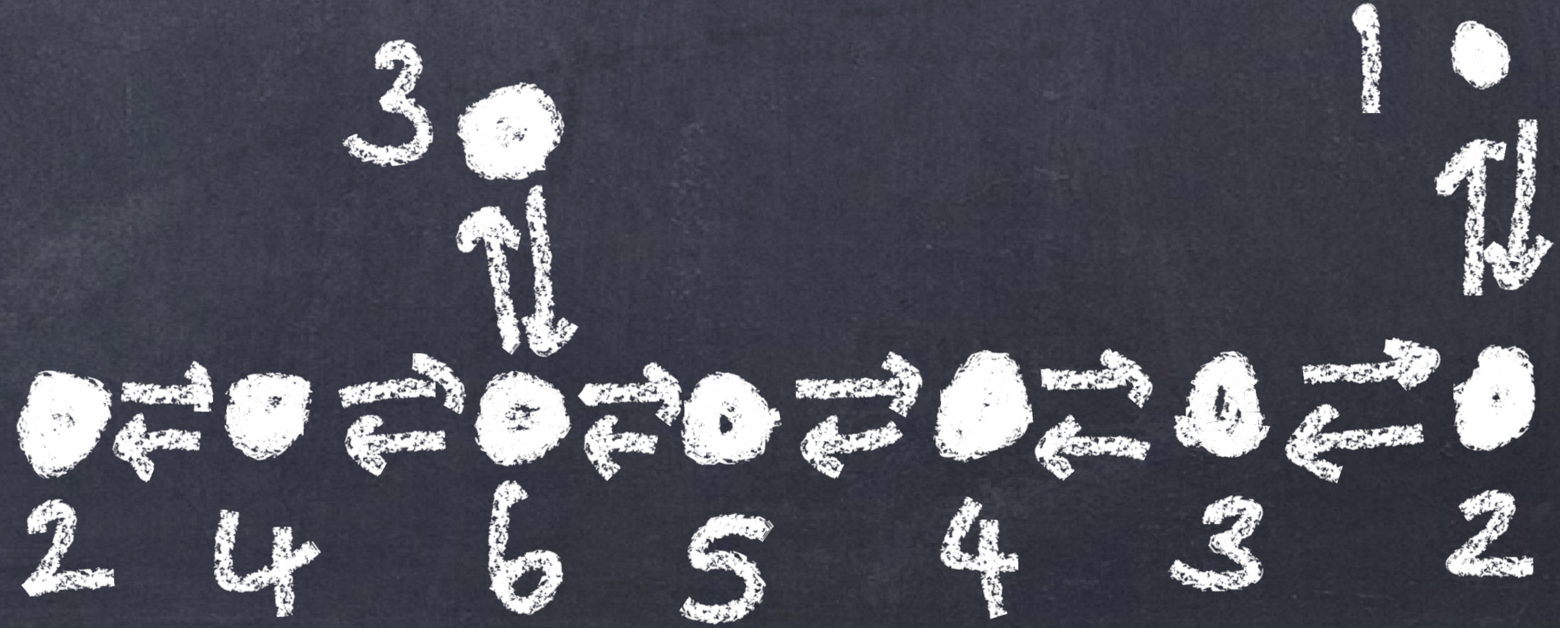

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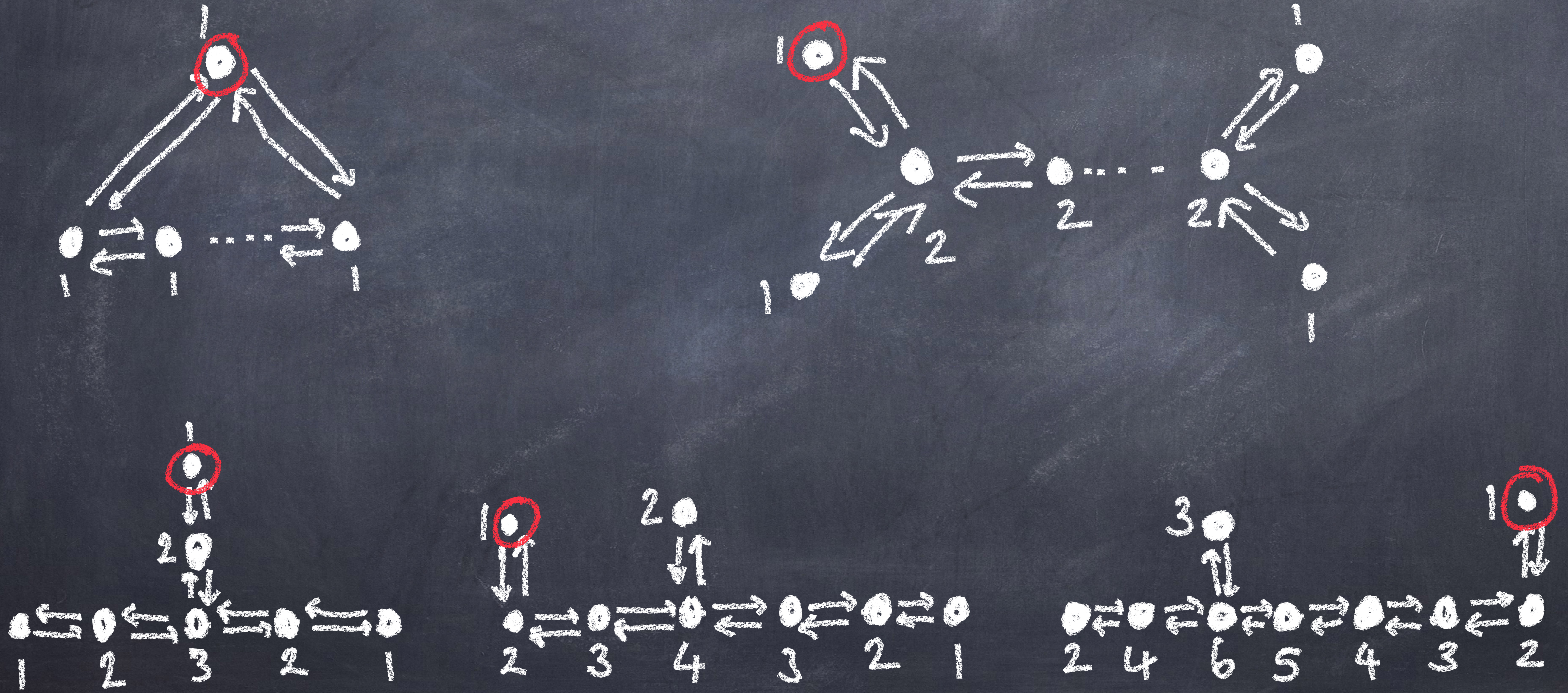

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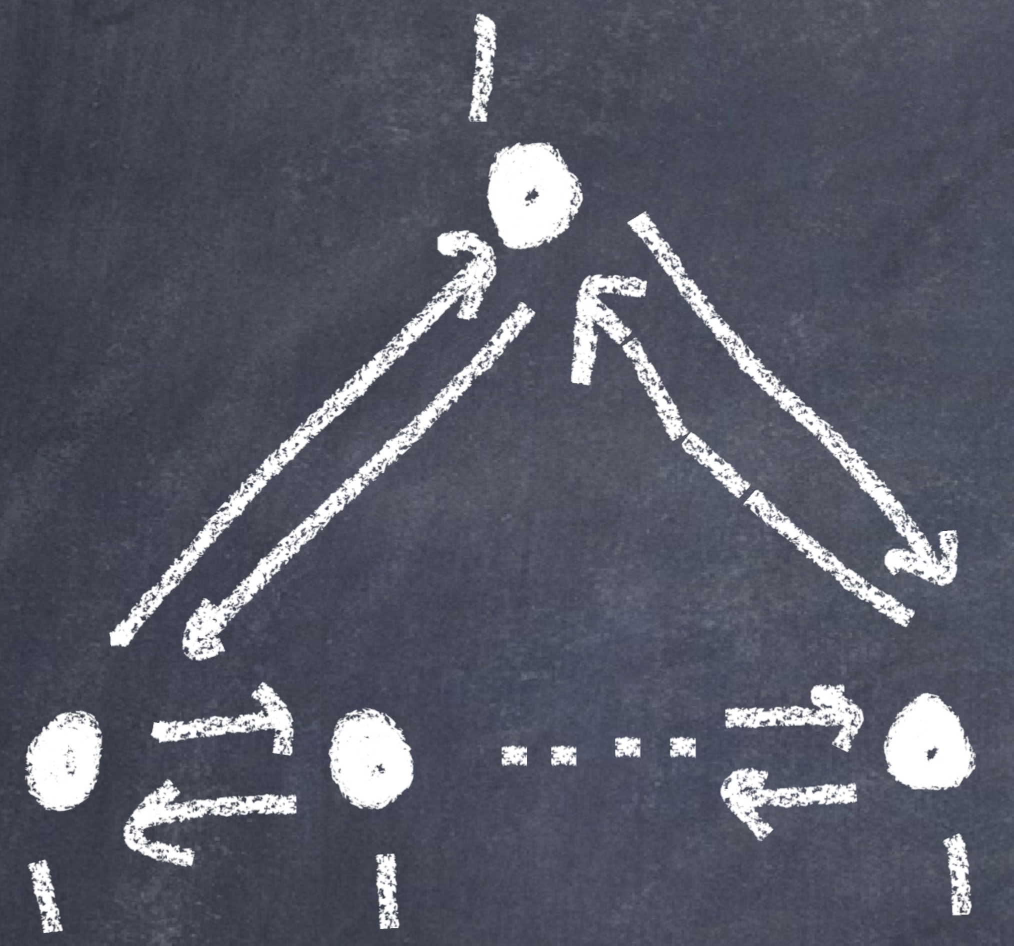
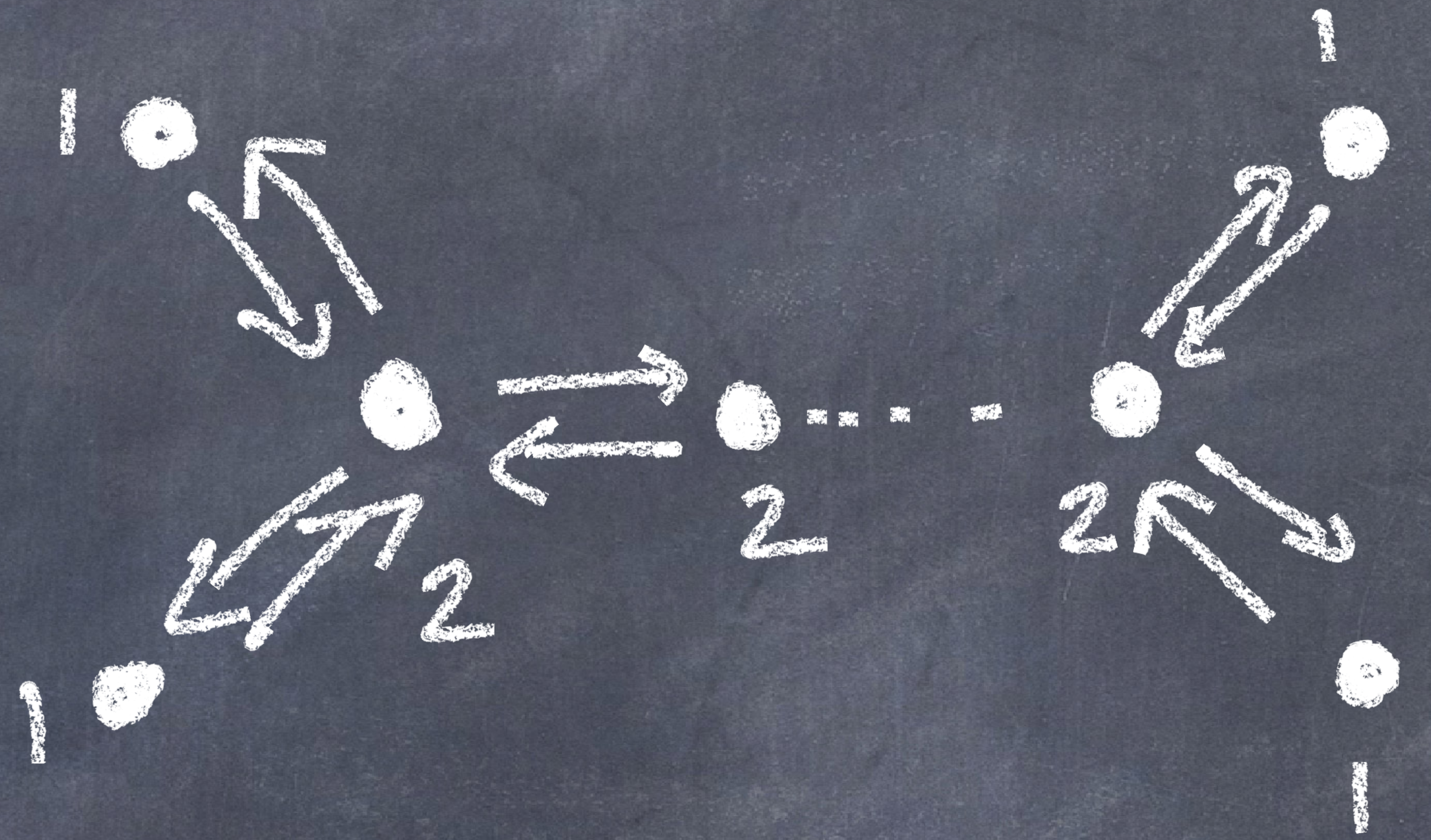
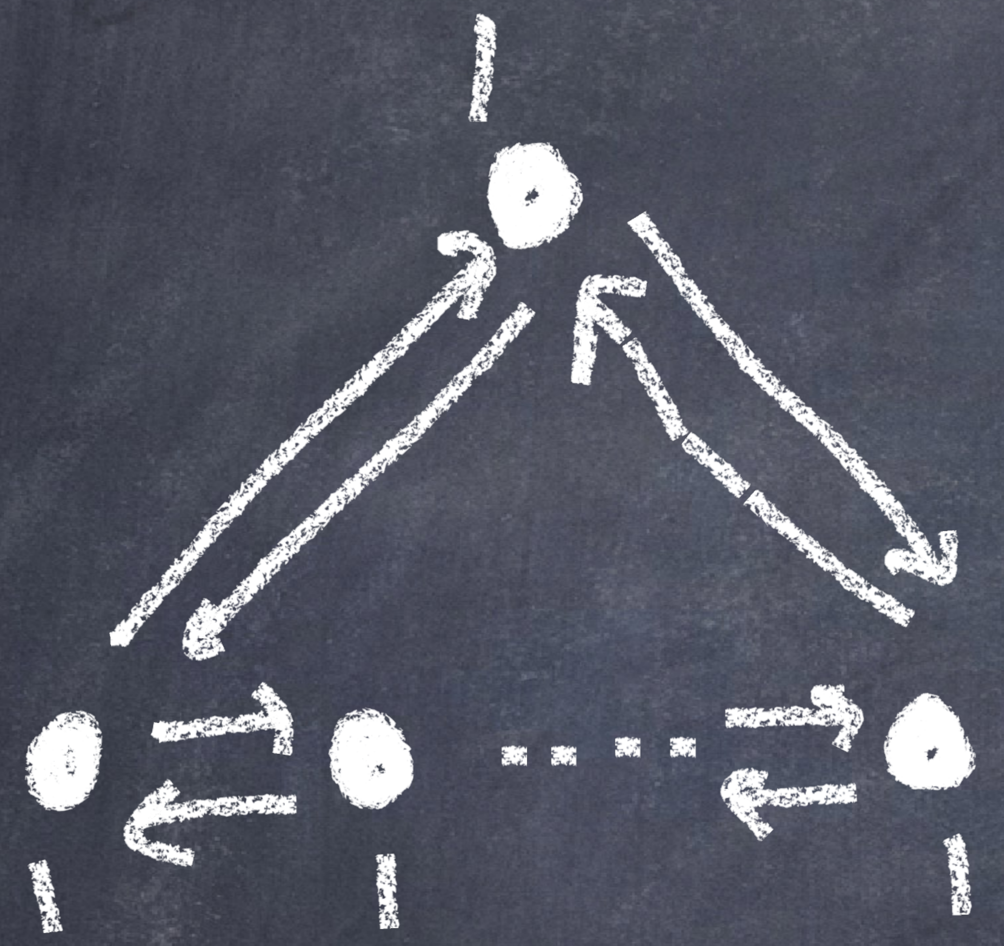


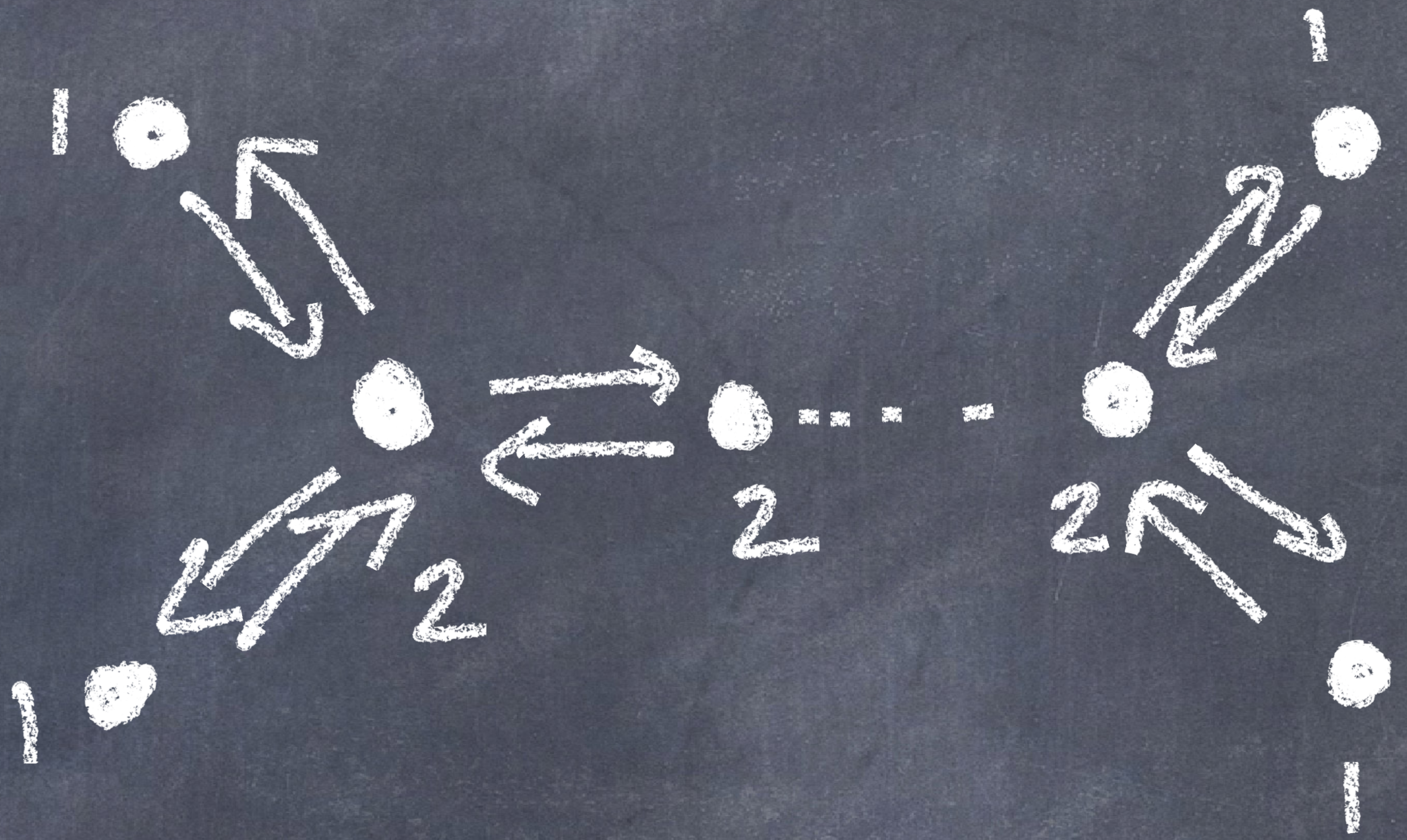


Representation of Q of dim vector δ = choice of linear map for every arrow in Q.

$$\mathrm{Rep}(Q,\delta) = \bigoplus_{a \in Q} \mathrm{Hom}\left(\mathbb{C}^{t(a)}, \mathbb{C}^{h(a)}\right)$$

Quiver varieties





Representation of Q of dim vector δ = choice of linear map for every arrow in Q.

$$\mathrm{Rep}(Q,\delta) = \bigoplus_{a \in Q} \mathrm{Hom}\left(\mathbb{C}^{t(a)}, \mathbb{C}^{h(a)}\right) = \text{fin. gen left } \Bbbk Q\text{-modules}$$

Quiver varieties

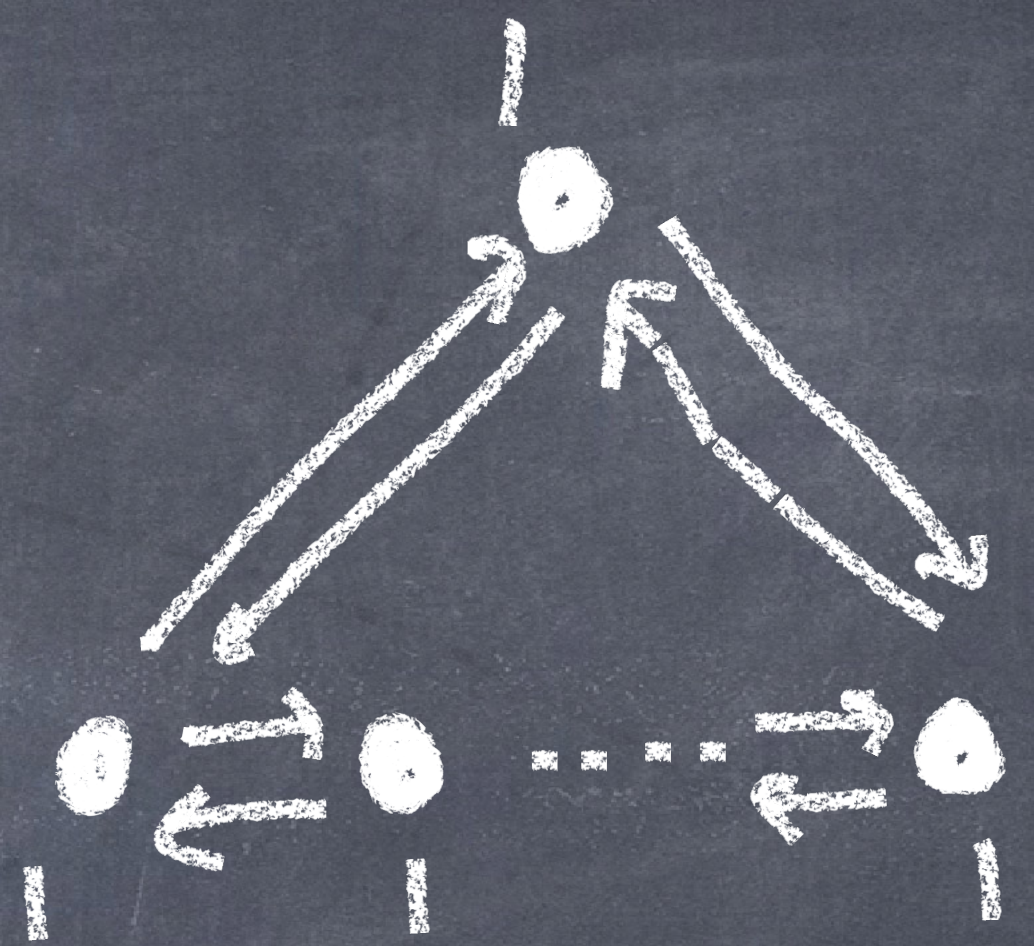
$\mathrm{Rep}(\Pi_\Gamma, v) \subset \mathrm{Rep}(Q, v)$

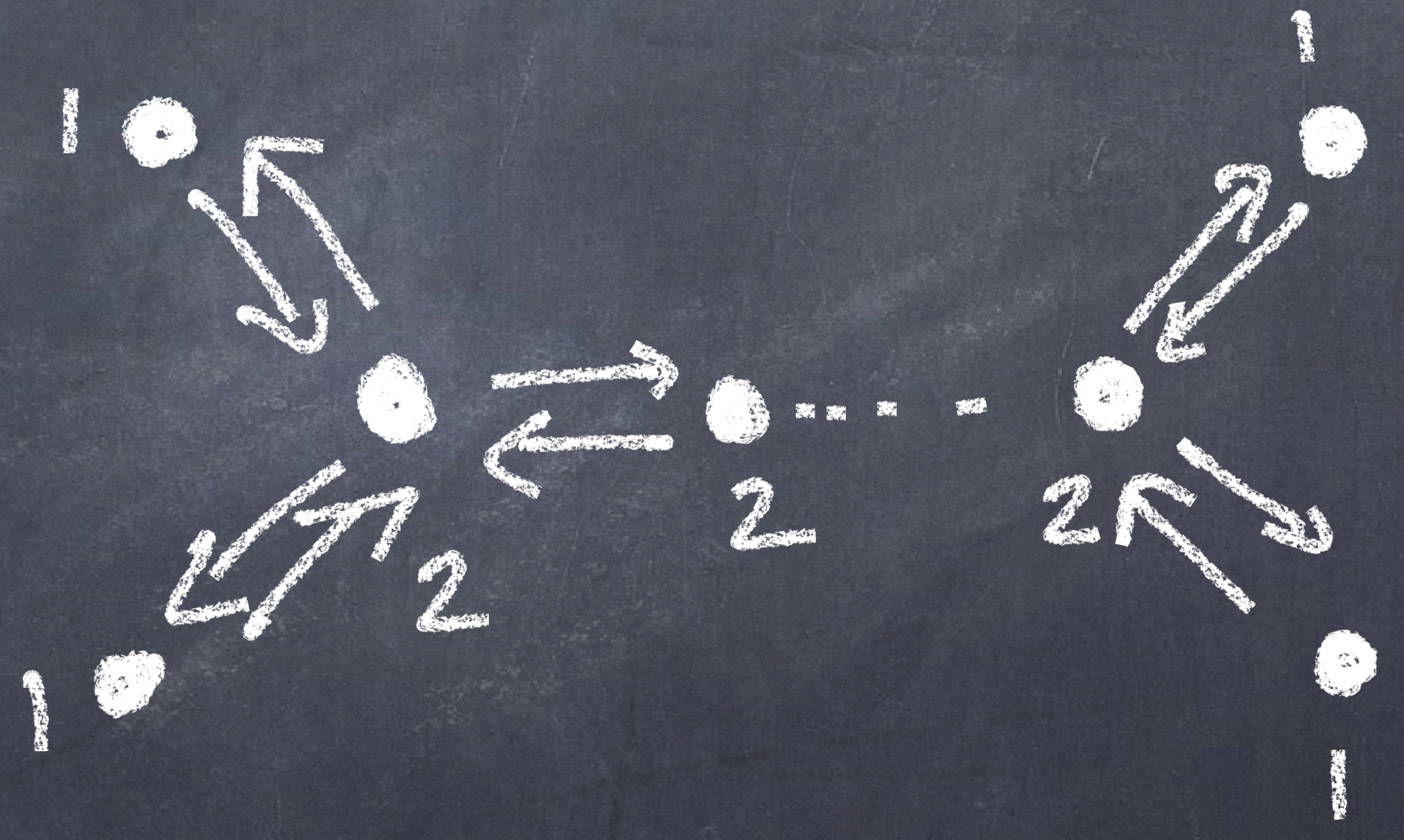
$\|$

Representations satisfying 'preprojective relations'

$\|$

Modules over $\Pi_\Gamma := \Bbbk Q/\sim$





Quiver varieties

Stability condition $\theta \in \mathbb{Z}^{r+1}$

Quiver variety $\mathcal{M}_\theta(\Pi_r, \delta)$: parametrises θ-semistable Π_r-modules.

$$\mathcal{M}_\theta(\Pi_r, \delta) \cong \operatorname{Rep}(\Pi_r, \delta) /\!\!/_\theta \, G(v)$$

Quiver varieties

Stability condition $\theta \in \mathbb{Z}^{r+1}$

$\rightsquigarrow$ Quiver variety $\mathcal{M}_\theta(\Pi_\Gamma, \delta)$: parametrises θ-semistable Π_Γ-modules.

- $\mathcal{M}_\theta$ is a Coarse moduli space
- If θ is generic, then $\mathcal{M}_\theta$ is a fine moduli space.

Why?

Why?

For generic θ, $\mathcal{M}_\theta(\Pi_\Gamma, \delta) \cong S$, a fine moduli space, so S has a tautological bundle $\mathcal{V}$.

Important : $\mathcal{V} = \bigoplus_{i \in Q} \mathcal{V}_i$, rank $\mathcal{V}_i = \dim \rho_i = \delta_i$

Normalise : $\mathcal{V}_0 \cong \mathcal{O}_S$

$\mathrm{End}(\mathcal{V}) \cong \Pi_\Gamma$

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For generic θ, $\mathcal{M}_\theta(\Pi_\Gamma, \delta) \cong S$, a fine moduli space, so S has a tautological bundle $\mathcal{V}$.

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Normalise : $\mathcal{V}_0 \cong \mathcal{O}_S$

$\operatorname{End}(\mathcal{V}) \cong \Pi_\Gamma$

$\mathcal{V}$ is a tilting bundle on S!

Tilting bundle

$\mathcal{V}$ a tilting object in $D^b(\mathrm{coh}\, S)$ means:

1. $\mathrm{Ext}^k_{\mathcal{O}_S}(\mathcal{V}, \mathcal{V}) = 0$ for $k \neq 0$
2. $\mathcal{V}$ generates $D^b(\mathrm{coh}\, S)$
3. $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{V}, -)$ commutes with direct sums.

Equivalence of categories

S projective over $\mathbb{C}^2/\Gamma$ (affine variety) $\rightsquigarrow$
Equivalence of derived categories

$$D^b(\mathrm{coh}\, S) \longleftrightarrow D^b(\mathrm{End}(\mathcal{V})\text{-mod})$$

given by $R\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{V}, -)$ and $- \overset{L}{\otimes} \mathcal{V}$

(Thm 7.6 of Hille - Van den Bergh '10)

Equivalence of categories

Side note :

Baer-Bondal theorem: similar, but for projective X, our case is slightly more subtle as S is projective over affine $\mathbb{C}^2/\Gamma$

What about S_K ?

- S_K not smooth for $K \neq \emptyset$.
- $S_K \cong \mathcal{M}_\xi(\Pi_r, \delta)$ but ξ <u>not</u> generic, so no tautological bundle, so no tilting bundle ...

What about S_K ?

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But there is a fix!

The fix - a different quiver

Idea: 'ignore' nodes of quiver corresponding to the contracted curves K.

Let $J = \{0, 1, \dots, r\} \setminus K$ (so $0 \in J$ always)

Consider $\mathcal{V}_J = \bigoplus_{j \in J} \mathcal{V}_j$

$\operatorname{End} \mathcal{V}_J \cong \Pi_r^J$

paths in Π_r starting and ending at nodes in J.

The fix - a different quiver

Idea : 'Ignore' nodes of quiver corresponding to the contracted curves K.

- Construct Q_J st $\Bbbk Q_J/_{\mathcal{I}} \cong \Pi_r^J$
  - ↳ Delete the nodes in K, add some extra arrows.
- New quiver variety $\mathcal{M}_\chi(\Pi_r^J, \delta_J)$

The fix – a different quiver

$S_K \cong \mathcal{M}_{\chi_J}(\Pi_\Gamma^J, \delta_J)$ (Craw-Yamagishi)

for χ_J a <u>generic</u> stability condition.

So $\exists$ a fine moduli space description of S_K, and a tautological bundle on S_K, $T_J = \bigoplus_{j \in J} T_j$

Normalise : $T_0 \cong \mathcal{O}_{S_K}$

The fix - a different quiver

Example: A_2 singularity, S_1

The fix - a different quiver

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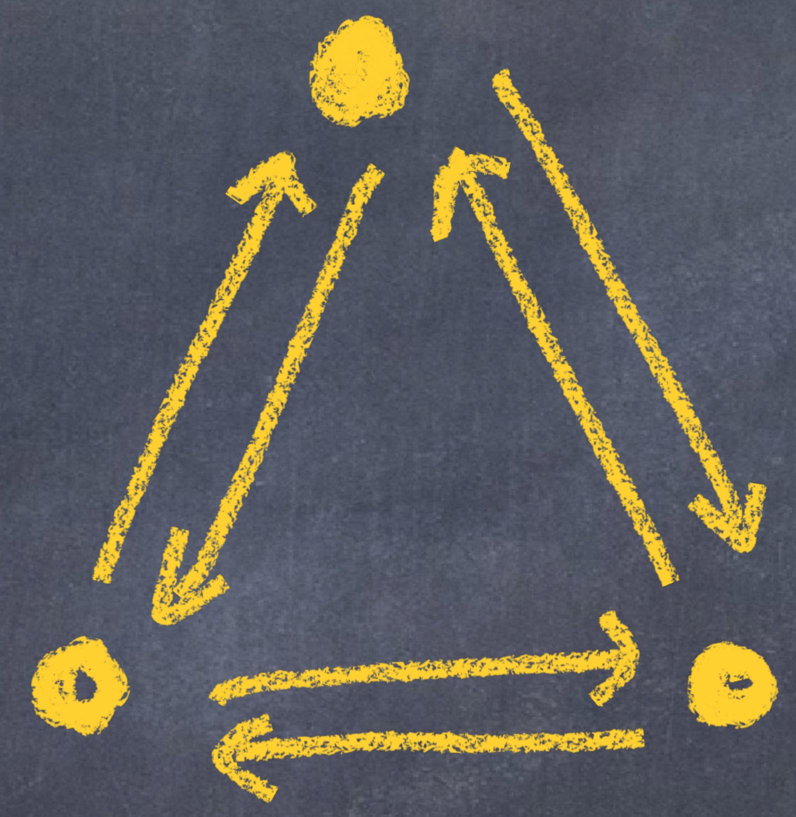
Original quiver :



The fix - a different quiver

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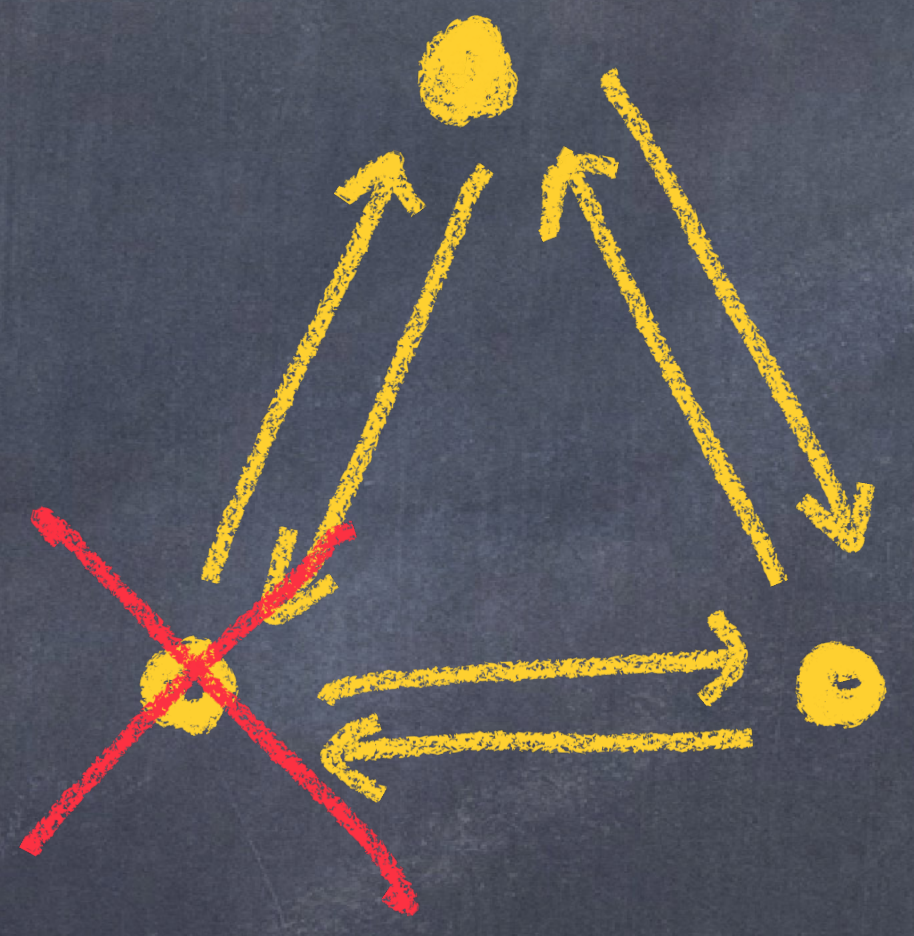

Ignore this vertex

The fix - a different quiver

Example: A_2 singularity, S_1

New quiver: Q_J

(possible some arrows are superfluous, possible there are some more).

The fix - a different quiver

Example: A_2 singularity, S_1

New quiver: Q_J

$\exists$ epimorphism $\Bbbk Q_J \to \mathrm{End}(\mathcal{V}_J)$ of $\Bbbk$-algebras

The fix - a different quiver

Example: A_2 singularity, S_1

New quiver: Q_J

$$S_1 \cong \mathcal{M}_{(-1,1)}\left(\mathbb{k}Q_J/\sim\,,(1,1)\right),\ T_J = T_0 \oplus T_2$$

Key question

Is T_J a tilting bundle?

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Is T_J a tilting bundle?

Does T_J play the same role as a tilting bundle and give us an equivalence of categories?

Dream scenario

An equivalence of categories

$$D^b(\mathrm{coh}(S_K)) \longleftrightarrow D^b(\mathrm{End}(\mathcal{V}_J)\text{-mod})$$

Putting together a proof

<u>Note</u> : $\mathrm{End}(\mathcal{V}_J)$-modules $\cong \mathrm{End}(T_J)$-modules.

Putting together a proof

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Helpful find? $Y \to X = \operatorname{Spec} R$ projective

Corollary 3.2.8. *Assume that $\mathcal{P}$ is a projective generator for ${}^p\operatorname{Per}(Y/X)$. Put $A = \operatorname{End}_Y(\mathcal{P})$ and write ${}_A\mathcal{P}$ to emphasize the left A-structure on $\mathcal{P}$ Then the functors $\operatorname{RHom}_Y({}_A\mathcal{P}, -)$ and $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ define inverse equivalences between $D^b(\operatorname{coh}(Y))$ and $D^b(A)$. These equivalences restrict to equivalences between ${}^p\operatorname{Per}(Y/X)$ and $\operatorname{mod}(A)$.*

Putting together a proof

Note: $\mathrm{End}(\mathcal{V}_J)$-modules $\cong$ $\mathrm{End}(T_J)$-modules.

Helpful find? $S_K \to \mathbb{C}^2/\Gamma = \mathrm{Spec}\, R$ projective

Corollary 3.2.8. *Assume that $\mathcal{P}$ is a projective generator for ${}^p\,\mathrm{Per}(Y/X)$. Put $A = \mathrm{End}_Y(\mathcal{P})$ and write ${}_A\mathcal{P}$ to emphasize the left A-structure on $\mathcal{P}$ Then the functors $\mathrm{RHom}_Y({}_A\mathcal{P}, -)$ and $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ define inverse equivalences between $D^b(\mathrm{coh}(Y))$ and $D^b(A)$. These equivalences restrict to equivalences between ${}^p\,\mathrm{Per}(Y/X)$ and $\mathrm{mod}(A)$.*

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<u>Note</u> : $\mathrm{End}(\mathcal{V}_J)$-modules $\cong$ $\mathrm{End}(T_J)$-modules.

Helpful find?

Is T_J a projective generator?

Corollary 3.2.8. *Assume that $\mathcal{P}$ is a* $\boxed{\textit{projective generator}}$ *for ${}^p\,\mathrm{Per}(Y/X)$. Put $A = \mathrm{End}_Y(\mathcal{P})$ and write ${}_A\mathcal{P}$ to emphasize the left A-structure on $\mathcal{P}$ Then the functors $\mathrm{RHom}_Y({}_A\mathcal{P}, -)$ and $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ define inverse equivalences between $D^b(\mathrm{coh}(Y))$ and $D^b(A)$. These equivalences restrict to equivalences between ${}^p\,\mathrm{Per}(Y/X)$ and $\mathrm{mod}(A)$.*

Putting together a proof

If $\mathcal{M}$ is a vector bundle of rank r on Y then by $c_1(\mathcal{M})$ we denote the class of $\wedge^r \mathcal{M}$ in $\mathrm{Pic}(Y)$.

Proposition 3.2.7. *The projective generators in* ${}^{-1}\,\mathrm{Per}(Y/X)$ *are the objects* $\mathcal{M}$ *in* $\mathfrak{V}$ *such that* $c_1(\mathcal{M})$ *is ample and such that* $\mathcal{O}_Y$ *is a direct summand of some* $\mathcal{M}^{\oplus a}$. *The projective generators in* ${}^{0}\,\mathrm{Per}(Y/X)$ *are the objects in* $\mathfrak{V}^*$ *which are dual to projective generators in* ${}^{-1}\,\mathrm{Per}(Y/X)$

So need
- $T_J \in \mathcal{B}$
- $c_1(T_J)$ ample
- $\mathcal{O}_{S_K}$ direct summand of some $T_J^{\oplus a}$

Putting together a proof

So need
- $T_J \in \mathcal{B}$
- $c_1(T_J)$ ample
- $\mathcal{O}_{S_K}$ direct summand of some $T_J^{\oplus a}$

$\mathcal{O}_{S_K} \cong T_0$, so $\mathcal{O}_{S_K}$ is a direct summand of T_J itself ✓

Putting together a proof

So need
- $T_J \in \mathcal{B}$
- $c_1(T_J)$ ample
- ~~$\mathcal{O}_{S_K}$ direct summand of some $T_J^{\oplus a}$~~

$c_1(T_j)$ = class of $\Lambda^{\dim \rho_j} T_j = \det T_j$

This is ample: it comes from an ample bundle on X_J-stable locus

Putting together a proof

So need $\bullet\ T_J \in \mathfrak{B}$?

~~$\bullet\ c_1(T_J)$ ample~~

~~$\bullet\ \mathcal{O}_{S_K}$ direct summand of some $T_J^{\oplus a}$~~

Let $\mathfrak{V}$ be the category of vector bundles $\mathcal{M}$ on Y generated by global sections such that $H^1(Y, \mathcal{M}^*) = 0$ and let $\mathfrak{V}^* = \{\mathcal{M}^* \mid \mathcal{M} \in \mathfrak{V}\}$.

Putting together a proof

So need • $T_J \in \mathfrak{B}$?

~~• $c_1(T_J)$ ample~~

~~• $\mathcal{O}_{S_K}$ direct summand of some $T_J^{\oplus a}$~~

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Need $H^1(S_K, T_J^\vee) = 0$, T_J gen by global sections

Putting together a proof

Need $H^1(S_K, T_J^\vee) = 0$, T_J gen by global sections

True as T_j is globally generated (Craw-Karmazyn-Ito, Prop 2.3).

Putting together a proof

Need $H^1(S_K, T_J^\vee) = 0$, T_J gen by global sections

More difficult.

True as T_j is globally generated (Craw-Karmazyn-Ito, Prop 2.3).

Final stage(s)

$$H'(S_K, T_J) = ?$$

Final stage(s)

$$H^1(S_K, T_J^\vee) \cong \mathrm{Ext}^1(\mathcal{O}_{S_K}, T_J^\vee)$$

$$\cong \mathrm{Ext}^1(T_J, \mathcal{O}_{S_K})$$

$$\cong \bigoplus_{j \in J} \underbrace{\mathrm{Ext}^1(T_j, \mathcal{O}_{S_K})}_{}$$

Simpler.

Final stage(s)

$$\mathrm{Ext}^1(T_j, \mathcal{O}_{S_K}) \cong \mathrm{Ext}^1(T_j, h_*(\mathcal{O}_S))$$

for $h : S \to S_K$ birational resolution,

S_K normal

Final stage(s)

$$\mathrm{Ext}^1(T_j, \mathcal{O}_{S_K}) \cong \mathrm{Ext}^1(T_j, h_*(\mathcal{O}_S))$$

for $h : S \to S_K$ birational resolution, S_K normal

$$\cong \mathrm{Ext}^1(h^*(T_j), \mathcal{O}_S)$$

by adjunction for h^* and h_*.

Final stage(s)

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for $h : S \to S_K$ birational resolution, S_K normal

$$\cong \mathrm{Ext}^1(h^*(T_j), \mathcal{O}_S)$$

(since $h^*(T_j) = \mathcal{V}_j$)

$$\cong \mathrm{Ext}^1(\mathcal{V}_j, \mathcal{V}_0)$$

Final stage(s)

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S_K normal

$$\cong \mathrm{Ext}^1(h^*(T_j), \mathcal{O}_S)$$

(since $h^*(T_j) = \mathcal{V}_j$)

$$\cong \mathrm{Ext}^1(\mathcal{V}_j, \mathcal{V}_0) = 0 \qquad \square .$$

Conclusion

Corollary 3.2.8. *Assume that $\mathcal{P}$ is a projective generator for ${}^p\operatorname{Per}(Y/X)$. Put $A = \operatorname{End}_Y(\mathcal{P})$ and write ${}_A\mathcal{P}$ to emphasize the left A-structure on $\mathcal{P}$ Then the functors $\operatorname{RHom}_Y({}_A\mathcal{P}, -)$ and $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ define inverse equivalences between $D^b(\operatorname{coh}(Y))$ and $D^b(A)$. These equivalences restrict to equivalences between ${}^p\operatorname{Per}(Y/X)$ and* $\operatorname{mod}(A)$.

So $\exists$ an equivalence of categories

$$D^b(\operatorname{coh}(S_K)) \longleftrightarrow D^b(\operatorname{End}(\mathcal{V}_J)\text{-mod}).$$

Historical note

- Known in the complete local case by Kalck-Iyama-Weymss-Yang.

Why do I care?

Considered $\mathrm{Hilb}^n(S_K)$.

↳ subschemes of length $n \subset S_K$, Z has structure sheaf $\mathcal{O}_Z \subset S_K$.

1. Use the derived equivalence to obtain a Π_ρ^J-module associated to $\mathcal{O}_Z$.

Prove particular properties of this module.

Why do I care?

Considered $\mathrm{Hilb}^n(S_K)$.

2. Use T_J to construct a bundle on $\mathrm{Hilb}^n(S_K)$, $T_J^{[n]} = p_* q^*(T_J)$ where p, q are projection maps from the universal subscheme $\tilde{\mathcal{Z}}_n$

Then describe the Nef and movable cone of $\mathrm{Hilb}^n(S_K)$ in terms of this bundle.

Thank you for listening!

Questions?