... a big fan of your heart.

7 NOVEMBER 2023

Setting: A is a finite dimensional algebra.
$H = \operatorname{mod} A$, $D = D^b \operatorname{mod} A$.

$\circ \rightrightarrows \circ$ (arrows a, b from 0 to 1): $k\langle e_0, a, b, e_1\rangle$, $e_0 a = a$, $ab = 0 \cdots$, $a e_1 = a$, $e_1 a = 0$

$e_1 \; e_2 \; e_3$
$a \; b$
ab

§ MOTIVATION

① $A = k \bullet \rightrightarrows \bullet$, then $D \cong D^b \operatorname{Coh} \mathbb{P}^1$ [Beilinson, Bondal]
Specifically we have $A = \operatorname{End}(t)$ for $t = \mathcal{O} \oplus \mathcal{O}(1)$
giving functors

$$\begin{tikzcd} \mathcal{O} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[loop left, "id"] & \mathcal{O}(1) \arrow[loop right, "id"] \end{tikzcd}$$

$$D^b \operatorname{Coh} \mathbb{P}^1 \xrightarrow{\sim} D^b \operatorname{mod} A$$
$$x \longmapsto R\operatorname{Hom}(t, x)$$
$$y \overset{L}{\otimes}_A t \longmapsfrom y$$

eg $\mathcal{O}$ corresponds to $0 \rightrightarrows \mathbb{C}$
$\mathcal{O}(-1)[1]$ … $\mathbb{C} \rightrightarrows 0$,
$\mathcal{O}_{[x:y]}$ … $\mathbb{C} \overset{x}{\underset{y}{\rightrightarrows}} \mathbb{C}$

$\mathcal{O} \oplus \mathcal{O}(-1)[1]$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[dl] & (\to 0 \to \mathcal{O} \xrightarrow{0} \mathcal{O}(-1) \to 0) \arrow[d] \arrow[l, dashed] \\ \mathcal{O} \arrow[u, dashed] & & \mathcal{O}(-1)[1] \end{tikzcd}$$

two cohomologies in $\mathbb{P}^1$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \overset{0}{\underset{0}{\rightrightarrows}} \mathbb{C} = A \arrow[d] \\ & \mathbb{C} \overset{\cong}{\underset{0}{\rightrightarrows}} \mathbb{C} \end{tikzcd}$$

single cohomology in A

② $A = k(\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet})$, $B = k(\bullet \to \bullet \leftarrow \bullet)$ then $D^b \operatorname{mod} A \cong D^b \operatorname{mod} B$ [Auslander, Platzeck, Reiten]. More generally for $A = kQ$, Q acyclic and $a \in Q_0$ a sink, take $t = \tau^{-1} \underset{0 \to 0 \to \mathbb{C}}{\underline{S(a)}} \oplus \bigoplus_{b \neq a} P(b)$ $(= e_b \cdot A)$

$\mathbb{C} \to 0 \to 0$
$\mathbb{C} \xrightarrow{1} \mathbb{C} \to 0$

$S(a) \to \_\_ \to \tau^{-1} S(a)$

Then $B := \operatorname{End}(t) = k(\mu_a Q)$, and we have
$\mu_a Q$: reflecting all arrows at a

$$D^b \operatorname{mod} B \xrightarrow{\sim} D^b \operatorname{mod} A$$
$$x \longmapsto \operatorname{Hom}(t, x)$$
$$y \otimes_A t \longmapsfrom y$$

This is a key ingredient in proving Gabriel's theorem.

Question: Given H, can we find all abelian G such that $D^b G \cong D^b H$?

$$\begin{tikzcd} D^b H \arrow[r, leftrightarrow, "\sim"] & D^b G \\ H \arrow[u, hook] & G \arrow[u, hook] \arrow[ul, dashed] \end{tikzcd}$$

§ HEARTS OF t-STRUCTURES

Say $H \subseteq D$ (a full subcategory; $\operatorname{Coh} \mathbb{P}^1$) is the heart of a bdd t-str if
① $\forall h, h' \in H$, $\operatorname{Hom}(h, h'[<0]) = 0$ — $\operatorname{Ext}^i(h, h') = \operatorname{Hom}_D(h, h'[i])$
② $\forall d \in D$, $\exists !$ a filtration

$$\begin{tikzcd} 0 \arrow[r] & d^1 \arrow[r] \arrow[d] & d^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & d^n = d \arrow[d] \\ & h_{i_1}[-i_1] & h_{i_2}[-i_2] & & h_{i_n}[-i_n] \end{tikzcd}$$

- $h_i \in H \; \forall i$
- $i_1 > i_2 > \ldots > i_n$

given $d \in D$ I can write it as a complex of things in H and then take cohomology

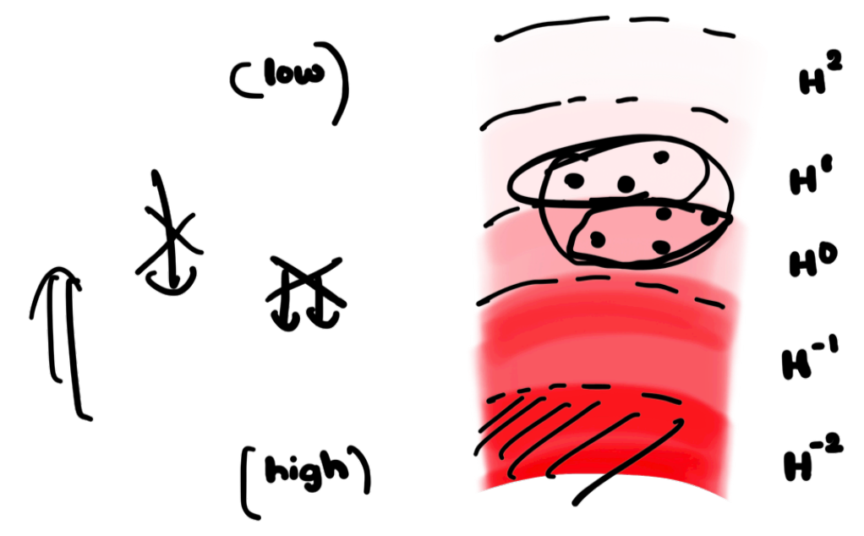
$$\begin{tikzcd} 0 \arrow[r] & (0 \to a \to 0 \to 0) \arrow[r] \arrow[d] & 0 \to a \to b \to 0 \arrow[d] \\ & (0 \to \underline{a} \to 0 \to 0) & (0 \to 0 \to \underline{b} \to 0) \end{tikzcd}$$

Definition. A pair $(H^{\leq 0}, H^{\geq 0})$ is a bounded t-structure [BBD]
if ① $\operatorname{Hom}(H^{\leq 0}, H^{\geq 1}) = 0$ where $H^{\geq 1} = H^{\geq 0}[1]$
② $H^{\geq 1} \subseteq H^{\geq 0}$
③ $\langle H^{\leq 0}, H^{\geq 1}\rangle$ is D
④ $D = \bigcup H^{\geq -n} \cap H^{\leq n} = \bigcup H^{[-n,n]}$
Say $H^0 = H^{\leq 0} \cap H^{\geq 0}$ is the heart.

$H^{\leq 0} = \langle H^0, H^{-1}, H^{-2}, H^{-3}, \ldots\rangle$

- Bounded t-structures are determined by the heart.

H^0 is automatically abelian:
$\underline{A \to B} \to C \to A[1]$
$\in H^0$ ⇕
$0 \to A \to B \to C \to 0$

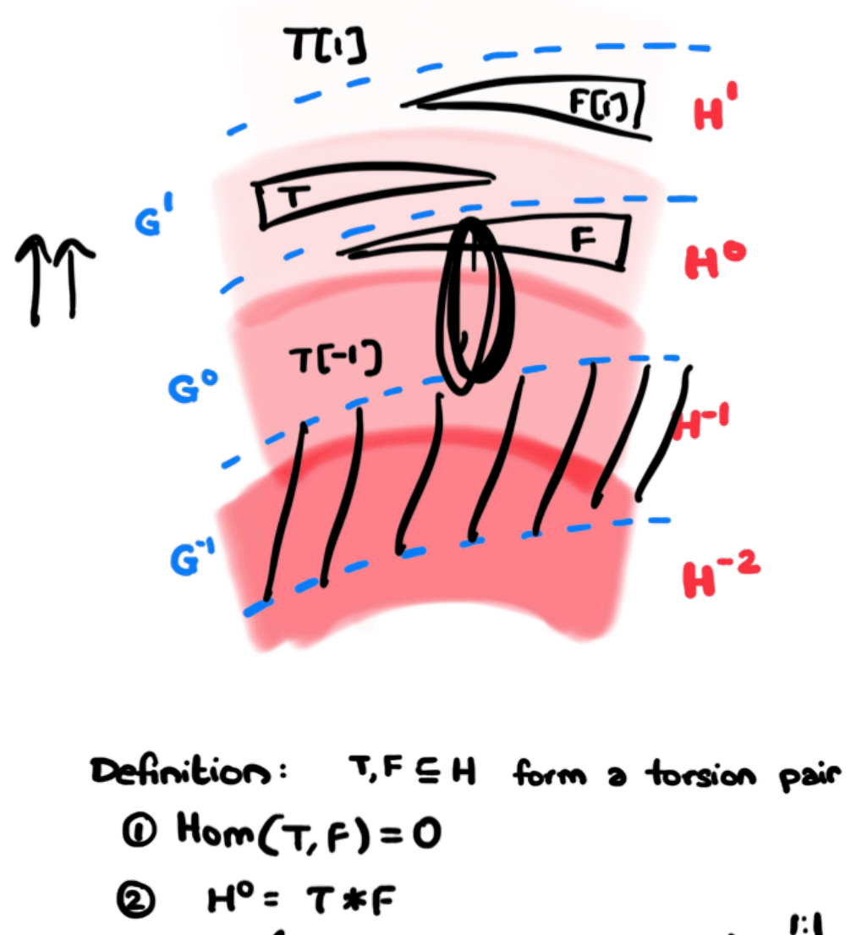


§ Intermediate t-structures

Happel Reiten Smalø

(Only matters up to shift.)

Say G is intermediate wrt H if $H^{\leq 0} \subseteq G^{\leq 0} \subseteq H^{\leq 1}$



$H^0 = \boxed{T * F} = \langle T, F\rangle$
$G^0 = F * T[-1]$

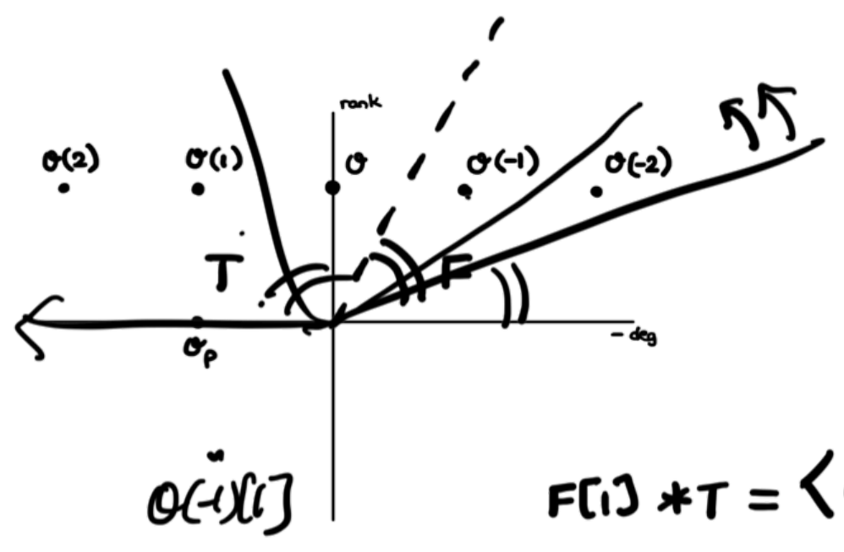
$H = \operatorname{Coh} \mathbb{P}^1$
T = torsion sheaves (skyscrapers, $\oplus$s)
F = free sheaves

Definition: $T, F \subseteq H$ form a torsion pair if
① $\operatorname{Hom}(T, F) = 0$
② $H^0 = T * F$

Thm (Torsion pairs in H) $\overset{1:1}{\longleftrightarrow}$ (Intermediate hearts)

Note just one of them is enough: $F = T^{\perp}$, ${}^{\perp}F = F$.
- In H noetherian, $(T, T^{\perp})$ is torsion pair
  $\Leftrightarrow$ T closed under quotients, extensions
- In H artinian, $({}^{\perp}F, F)$ is a torsion pair
  $\Leftrightarrow$ F closed under subs, extensions.

Example: $\operatorname{Coh} \mathbb{P}^1$



$F[1] * T = \langle \underline{\mathcal{O}(-1)[1]}, \underline{\mathcal{O}(-2)[1]}, \ldots,$
$\mathcal{O}, \underline{\mathcal{O}(1), \mathcal{O}(2), \ldots},$
$\mathcal{O}_p \rangle$

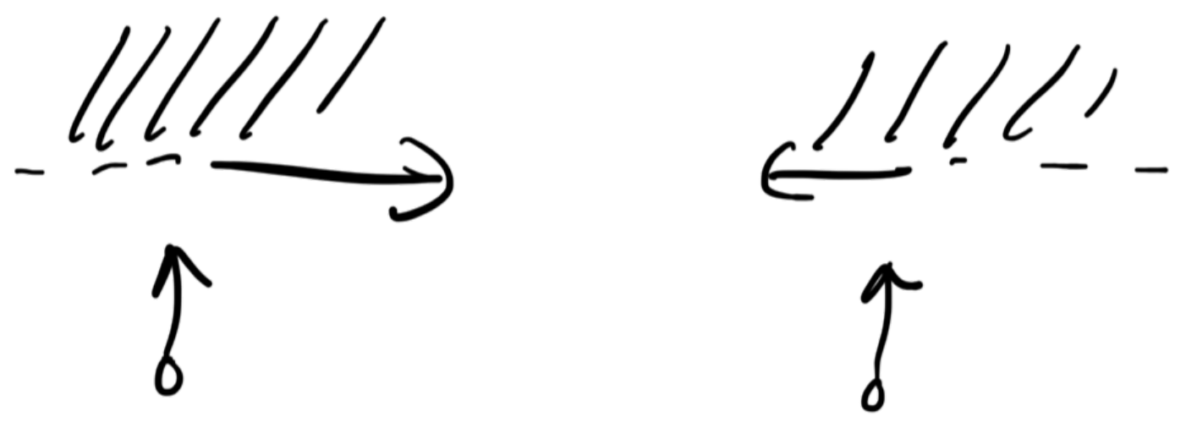
Mixed coherent hearts:

$F[1] * T = \langle \underset{\mathbb{C} \rightrightarrows 0}{\mathcal{O}(-1)[1]}, \underset{0 \rightrightarrows \mathbb{C}}{\mathcal{O}} \rangle$
$= \operatorname{mod} k \bullet \rightrightarrows \bullet$

$\langle \mathcal{O}(-n)[1], \mathcal{O}(-n+1)\rangle$
$\bullet \rightrightarrows \bullet$

$\operatorname{Tor} = \operatorname{Sky}$

Lb, $\operatorname{Sky}[1]$



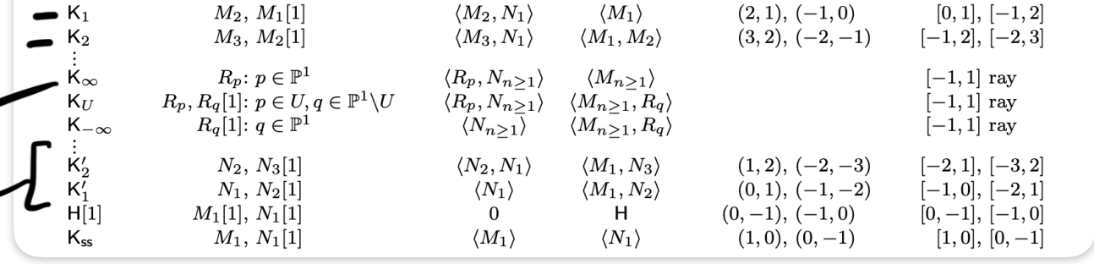
§ THE HEART FAN

The Grothendieck group of D is
$KD = \underline{\mathbb{Z}\langle h \mid h \in D\rangle}$
$h = h' + h''$ wherever $h' \to h \to h'' \to h'[1]$

For $H \subseteq D$, define
Effective cone $EH = \mathbb{Z}_{\geq 0} \cdot \langle h \mid h \in H\rangle$
Dual cone $CH = \{ v : KD \to \mathbb{R} \mid v \geq 0 \text{ on } EH\}$

Proposition [Paukstello, Broomhead, Ploog, Woolf] → 2023 The ♡-fan of an abelian category
$\underline{\underline{\Sigma H}} = \{ \tau \leq CK \mid K \text{ an intermediate heart wrt } H\}$ is a fan in $KD^{\vee}_{\mathbb{R}}$
More precisely, the cones are given by
$C(K/S) = C(K) \cap S^{\perp}$ for $\underline{\underline{S}} \subset K$ a face subcategory

$R_p = \mathcal{O}_p$, $p \in \mathbb{P}^1$
$N_i = \mathcal{O}(-i)[1]$
$M_i = \mathcal{O}(i)$
$H = \bullet \rightrightarrows \bullet$

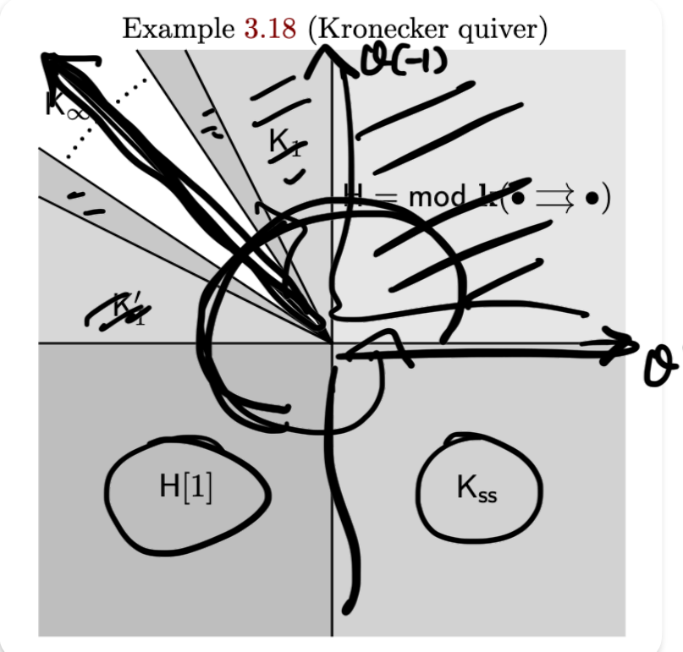


heart K	simple objects	T	F	$E(\mathsf{K})$	$C(\mathsf{K})$
H	M_1, N_1	H	0	(1, 0), (0, 1)	[1, 0], [0, 1]
K_1	M_2, $M_1[1]$	$\langle M_2, N_1\rangle$	$\langle M_1\rangle$	(2, 1), (−1, 0)	[0, 1], [−1, 2]
K_2	M_3, $M_2[1]$	$\langle M_3, N_1\rangle$	$\langle M_1, M_2\rangle$	(3, 2), (−2, −1)	[−1, 2], [−2, 3]
⋮					
K_∞	R_p: $p \in \mathbb{P}^1$	$\langle R_p, N_{n\geq 1}\rangle$	$\langle M_{n\geq 1}\rangle$		[−1, 1] ray
K_U	$R_p, R_q[1]$: $p \in U, q \in \mathbb{P}^1\backslash U$	$\langle R_p, N_{n\geq 1}\rangle$	$\langle M_{n\geq 1}, R_q\rangle$		[−1, 1] ray
$\mathsf{K}_{-\infty}$	$R_q[1]$: $q \in \mathbb{P}^1$	$\langle N_{n\geq 1}\rangle$	$\langle M_{n\geq 1}, R_q\rangle$		[−1, 1] ray
⋮					
K'_2	N_2, $N_3[1]$	$\langle N_2, N_1\rangle$	$\langle M_1, N_3\rangle$	(1, 2), (−2, −3)	[−2, 1], [−3, 2]
K'_1	N_1, $N_2[1]$	$\langle N_1\rangle$	$\langle M_1, N_2\rangle$	(0, 1), (−1, −2)	[−1, 0], [−2, 1]
H[1]	$M_1[1]$, $N_1[1]$	0	H	(0, −1), (−1, 0)	[0, −1], [−1, 0]
K_{ss}	M_1, $N_1[1]$	$\langle M_1\rangle$	$\langle N_1\rangle$	(1, 0), (0, −1)	[1, 0], [0, −1]

$\bullet \rightrightarrows \bullet$
$\mathbb{P}^1$
$\bullet \rightrightarrows \bullet$

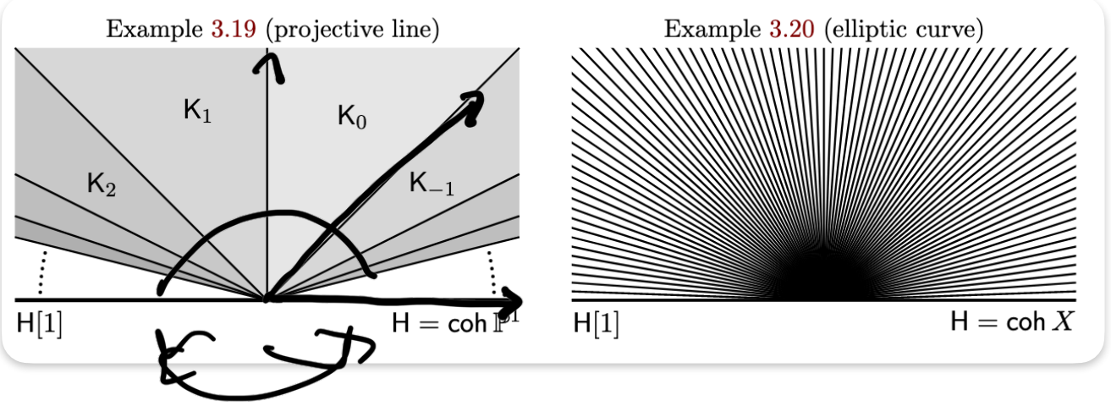
H H[1]

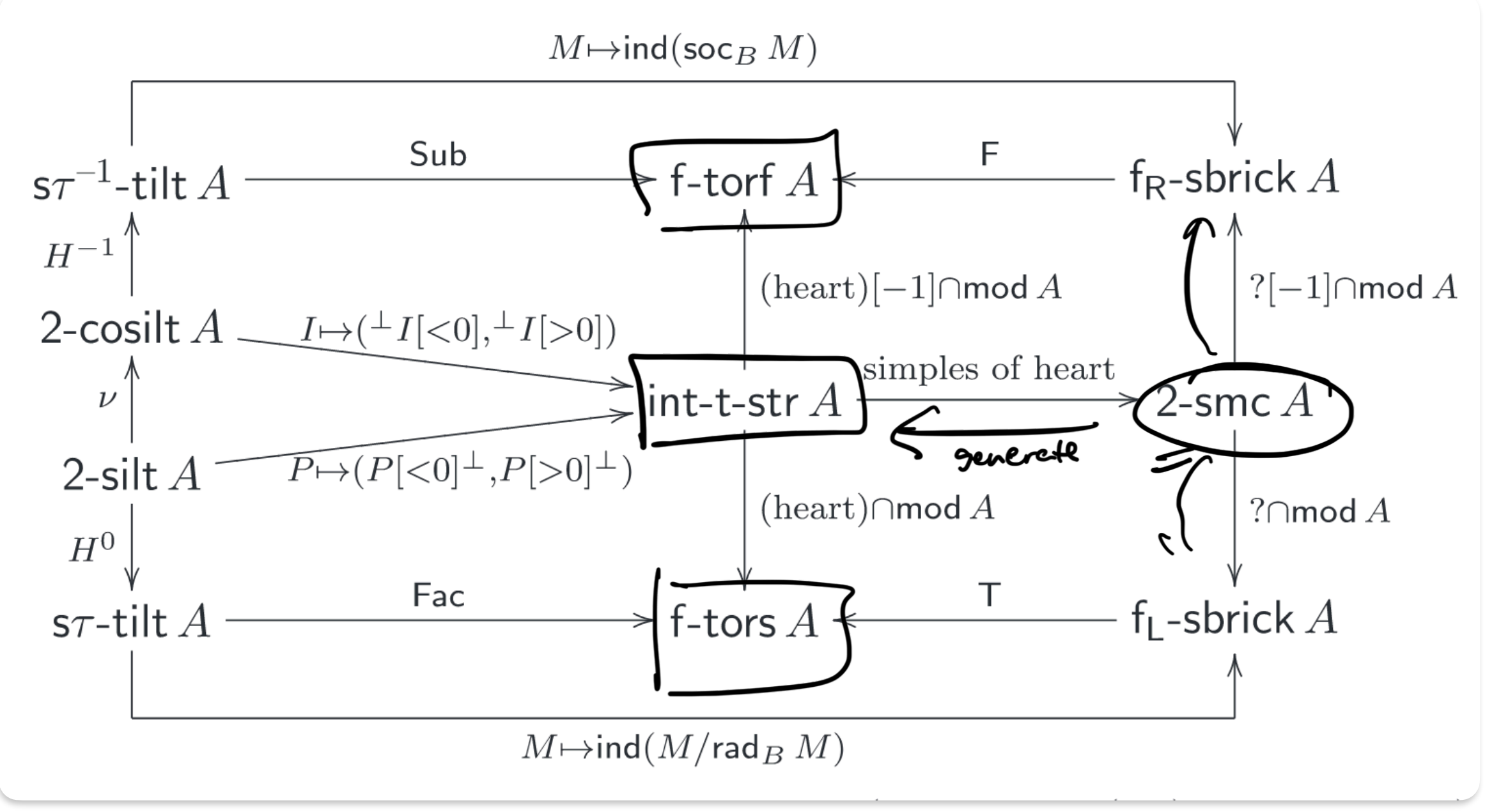
K K[1]


Example 3.18 (Kronecker quiver)

$$\begin{tikzcd} \operatorname{Coh} \arrow[d] & & & \\ D^b \operatorname{mod} \bullet\rightrightarrows\bullet \arrow[r, "\sim"] & D^b \operatorname{mod}(\bullet\rightrightarrows\bullet) \arrow[r, "\sim"] & D^b \operatorname{mod}(\bullet\rightrightarrows\bullet) \\ & \underline{\operatorname{mod} \bullet\rightrightarrows\bullet} \arrow[u] & \underline{\operatorname{mod} \bullet\rightrightarrows\bullet} \arrow[u] \end{tikzcd}$$

$\langle \underline{\mathcal{O}}, \mathcal{O}(-1)[1]\rangle$ $\langle \underline{\mathcal{O}(2)}, \underline{\mathcal{O}(1)[1]}\rangle$


Example 3.19 (projective line) Example 3.20 (elliptic curve)



$$\begin{tikzcd}[column sep=large] s\tau^{-1}\text{-tilt } A \arrow[r, "\mathsf{Sub}"] \arrow[rr, bend left, "M \mapsto \mathsf{ind}(\mathsf{soc}_B M)"] & \boxed{\text{f-torf } A} & \text{f}_{\mathsf{R}}\text{-sbrick } A \arrow[l, "\mathsf{F}"'] \\ \text{2-cosilt } A \arrow[u, "H^{-1}"] \arrow[r, "I \mapsto ({}^{\perp}I[<0], {}^{\perp}I[>0])"] & \boxed{\text{int-t-str } A} \arrow[u, "(\text{heart})[-1] \cap \mathsf{mod}\, A"'] \arrow[d, "(\text{heart}) \cap \mathsf{mod}\, A"] \arrow[r, "\text{simples of heart}"] & \text{2-smc } A \arrow[u, "?[-1] \cap \mathsf{mod}\, A"'] \arrow[d, "? \cap \mathsf{mod}\, A"] \arrow[l, bend left, "\text{generate}"] \\ \text{2-silt } A \arrow[u, "\nu"] \arrow[d, "H^0"] \arrow[ru, "P \mapsto (P[<0]^{\perp}, P[>0]^{\perp})"'] & & \\ s\tau\text{-tilt } A \arrow[r, "\mathsf{Fac}"] \arrow[rr, bend right, "M \mapsto \mathsf{ind}(M/\mathsf{rad}_B M)"'] & \boxed{\text{f-tors } A} & \text{f}_{\mathsf{L}}\text{-sbrick } A \arrow[l, "\mathsf{T}"'] \end{tikzcd}$$

Asai, Semibricks, 2020