A (blinkered) tour of the Mckay correspondence.

1 Mckay's observation (1980)

Let $V \cong \mathbb{C}^2$. $G \subset SL(V)$ finite.

Via representation theory we get a quiver.

vertices : $\rho_i \in \mathrm{Irreps}(G)$

d_{ij} = #edges $\rho_i \to \rho_j$

$$d_{ij} = \dim\left(\mathrm{Hom}_G(\rho_i \otimes V^*, \rho_j)\right)$$

we call this the Mckay quiver.

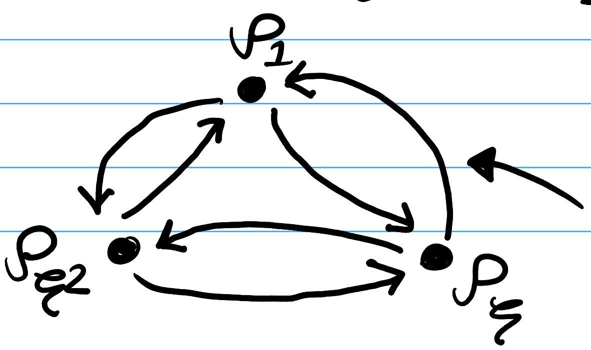
ex $\xi^3 = 1$, $\xi \neq 1$

$$G = \left\langle g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix} \right\rangle \cong \mathbb{Z}/3\mathbb{Z}$$

irreps of G : ρ_1, ρ_ξ, ρ_{ξ^2}

$V^* \cong \rho_{\xi^2} \oplus \rho_\xi$

$\rho_\xi \otimes V^* \cong \rho_1 \oplus \rho_{\xi^2}$



Q_G

There is geometric side :

Take quotient V/G.

Take minimal resolution

$$f : X \longrightarrow V/G.$$

$$f^{-1}(0) = \bigcup_i E_i \,, \quad E_i \cong \mathbb{P}^1$$

create quiver Q_X.

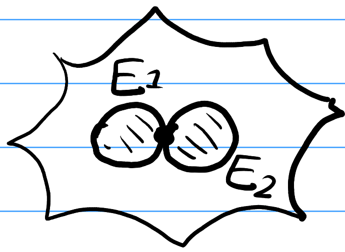
vertices : E_i

$f_{ij} = \#\text{edges } E_i \to E_j$

$$f_{ij} = \begin{cases} E_i \cdot E_j \,, & i \neq j \,, \\ 0 \,, & i = 0 \,. \end{cases}$$

<u>ex</u> take $G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix} \right\rangle$.

$$\begin{array}{c} V/G = (xy = z^3) \\ \Big\uparrow f \\ X \end{array} \quad = \quad$$



$$Q_X = \bullet \rightleftarrows \bullet$$

McKay's observation :

Take $Q_G \rightsquigarrow \widetilde{Q_G}$ by removing ρ_1

then $\widetilde{Q_G} = Q_E$.

$G = \left\langle \begin{pmatrix} q & 0 \\ 0 & q^2 \end{pmatrix} \right\rangle$

$Q_G =$  $\qquad \bullet \rightleftarrows \bullet = Q_X.$

$V \cong \mathbb{C}^2$

$$\begin{tikzcd} G \subset SL(V) \text{ finite.} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, "\tilde{Q}_G"'] & \text{ADE classification} & V/G \text{ du Val / Klein singularities ADE} \\ & \text{Dynkin diagrams} \arrow[ur, leftrightarrow, "Q_X"'] & \end{tikzcd}$$

2. Objects

throughout: $V \cong \mathbb{C}^n$

$G \subset SL(V)$ finite

$f : X \to V/G$ will be a crepant res.

$f^* K_{V/G} \sim K_X$

We will want G-equivariant coh. sheaves on V.

$F \in \mathrm{Coh}^G(V)$

is just $F \in \mathrm{Coh}(V)$

together with isomorphisms

$\forall g \in G, \ \lambda_g : F \xrightarrow{\sim} g^* F$ ⟵ acting by g.

$+ \quad g^* \lambda_h \circ \lambda_g = \lambda_{gh}$

$+ \quad \lambda_{1_G} = 1_F.$

$\mathrm{Coh}^G(V)$ is an abelian cat.

$\rightsquigarrow D^G(V).$

3. Lost and found (in the resolution).

Ideal situation

H a finite group acts freely on Y (q.proj.)
there is an equivalence of categories.

$$\mathrm{Coh}(Y/H) \underset{\pi_*(-)^H}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Coh}^H(Y) \quad (\simeq)$$

V
$\downarrow$
V/G
$\cong$
Spec(inv. ring)

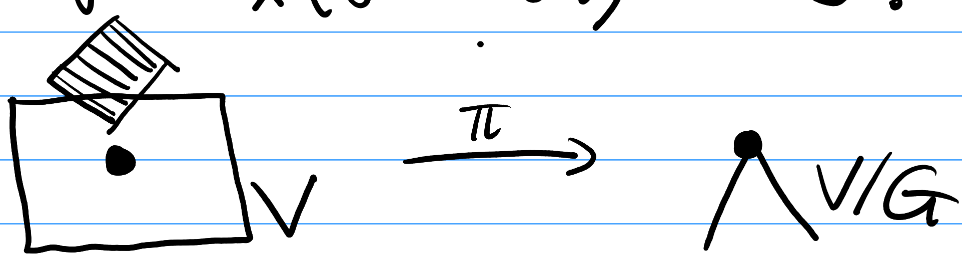
Unfortunately this almost never holds in our situation.
$(G \subset SL(V),\ G \curvearrowright V)$.

Take ρ an irrep of G (non-trivial).

$\rightsquigarrow \rho \otimes k(0) \in \mathrm{Coh}^G(V)$.

$k(0)$: skyscraper sheaf over 0.

ρ $\quad \pi_*(\rho \otimes k(0))^G = 0.$



"information about stabilisers being lost"

Remedy: replace V/G with a crepant resolution.

"a crepant resolution lie on the same level at $V \curvearrowleft G$"

$$\begin{tikzcd} X \arrow[r, dashed, leftrightarrow] \arrow[dr] & V \curvearrowleft G \arrow[d] \\ & V/G \end{tikzcd}$$

Theorem

If $n=2$ There is a triangulated equivalence

$$\Phi : D(X) \xrightarrow{\sim} D^G(V)$$

→ [Gonzalez-Sprinberg, Verdier]

$\Downarrow$ If you work hard

McKay's observation.

There is a deeper explaination...

4. Modularity of the resolution

$G \curvearrowright V$, away from the origin G acts freely.

$V/G - \underline{0}$ parameterise(s) free orbits in V.

Let $m = |G|$.

There is a map

$$(V/G - 0) \longrightarrow \mathrm{Hilb}^{[m]} V$$

parameterised $Z \subset V$ closed subschemes s.t. $H^0(Z, \mathcal{O}_Z)$ is m-dim'l.

We can X to be the closure of this.

In the case $n = 2, 3$.

$$X = G\text{-}\mathrm{Hilb}(V)$$

$H^0(Z, \mathcal{O}_Z) \cong \mathbb{C}[G]$ as a rep. of G.

There is a map $X \longrightarrow V/G$

this is a crepant resolution.
$n=3$ [Bridgeland-King-Reid].

UPSHOT * there exists a crepant resolution $= G\text{-}\mathrm{Hilb}(V)$.
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<u>Theorem</u>
If Z is the universal family on $G\text{-}\mathrm{Hilb}(V)$

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G\text{-}\mathrm{Hilb}(V) = X & & V \end{tikzcd}$$

then $\Phi := Rq_* \circ p^*$ gives an equivalence

$$D(X) \xrightarrow{\sim} D^G(V).$$

$$Z \subset X \times V$$

$$\begin{tikzcd}
Z \arrow[r] & X \\
Z_x \subset V \arrow[u, hook] \arrow[r] & x \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$X = G\text{-Hilb}(V)$$

$\ni$ $Z_x \subset V$ s.t.

$H^0(Z_x, \mathcal{O}_{Z_x}) \cong \mathbb{C}[G]$

$$\begin{tikzcd}
Z \arrow[r] & X \\
Z_x \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & x \arrow[u, hook]
\end{tikzcd}$$

"G-constellation"

$F \in \mathrm{Coh}^G(V)$, $H^0(V, F) \cong \mathbb{C}[G]$.

Theorem [2022] $n=3$

$\rightsquigarrow$ all crepant resolutions

$$X \longrightarrow V/G$$

are moduli spaces of G-constellations.

In addition

$\exists \quad D(X) \xrightarrow{\sim} D^G(V)$

& if X' is another crepant resolution then there is an equivalence

$$D(X) \longrightarrow D(X'). \quad \square$$

Ryo Yamagishi 2022.
"G-constellations" I & II

$\boxed{n=2}$

$$D(X) \longleftarrow D^G(V) : \Psi$$

If ρ_i is an irrep. (non-trivial)

$$\Psi(\rho_i \otimes K(0)) = \mathcal{O}_{E_i}(-1)[1].$$