Derived equivalences of holomorphic symplectic varieties

§ Motivation

X, Y sm proj var / $\mathbb{C}$

Thm (Orlov) $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$ induces an isom

$$\Phi^H : H^\bullet(X;\mathbb{Q}) \xrightarrow{\sim} H^\bullet(Y;\mathbb{Q})$$

- Φ^H respects the columns of the Hodge diamond.

⚠ Φ^H does <u>not</u> respect

the grading.

Conj: If $\Phi: D^b(X) \xrightarrow{\sim} D^b(Y)$
then $\exists$ isom
$H^\cdot(X;\mathbb{Q}) \xrightarrow{\sim} H^\cdot(Y;\mathbb{Q})$
that respects both
the grading & HS.

Known:

- X, Y (anti)-ample canonical bdl.
- Abelian (Mukai)
- X, Y hyperkähler (Taelman)

Thm (L) X, Y sm proj var / $\mathbb{C}$, admitting a holomorphic symplectic form and satisfying (*). If $D^b(X) \simeq D^b(Y)$ then $\exists$ isom $H^\bullet(X;\mathbb{Q}) \xrightarrow{\sim} H^\bullet(Y;\mathbb{Q})$ respecting grading + HS.

LLV algebra

Hard Lefschetz thm

X sm proj var $/\mathbb{C}$ of dim n. Then $\exists\, \omega \in H^2(X;\mathbb{Q})$ st multiplication by

$$\omega^k : H^{n-k}(X;\mathbb{Q}) \xrightarrow{\sim} H^{n+k}(X;\mathbb{Q})$$

is an isom $\forall\, k \leq n$.

Let $e_\omega : H^\bullet(X) \to H^\bullet(X)$ be $\alpha \mapsto \omega \cup \alpha$

Jacobson-Morozov Lemma

$\exists f_\omega : H^\bullet(X) \to H^\bullet(X)$

st $f_\omega^k : H^{n+k}(X) \to H^{n-k}(X)$

is the inverse of e_ω^k.

Furthermore

$\langle e_\omega, h, f_\omega \rangle \cong \mathfrak{sl}_2(\mathbb{Q})$

$h : H^\bullet(X) \to H^\bullet(X)$

$\alpha \in H^{n+k}(X)$

$\Rightarrow h\alpha = k\alpha$

~~Def~~ The LLV algebra

(Looijenga-Lunts-Verbitsky

is the Lie subalgebra of $\mathrm{End}(H^{\bullet}(X;\mathbb{Q}))$ generated by all $\mathfrak{sl}_2$-triples $\langle e_\omega, h, f_\omega\rangle$ where ω satisfies the HL thm.

<u>Notation:</u> $\mathfrak{g}(X;\mathbb{Q})$

<u>Ex</u> • $X = \mathbb{P}^n \quad n \geq 1$

$$H^i(X;\mathbb{P}^n) = \begin{cases} 0 & \text{else} \\ \mathbb{Q} & i \text{ even}, \; 0 \leq i \leq 2n \end{cases}$$

$$\Rightarrow \mathfrak{g}(\mathbb{P}^n;\mathbb{Q}) = \mathfrak{sl}_2(\mathbb{Q})$$

- X ab var, $V = H^1(X;\mathbb{Q})$
$\mathfrak{g}(X) = \mathfrak{so}(V \oplus V^\vee)$.
$\mathfrak{g}(X) \curvearrowright H^\bullet(X) = \Lambda^\bullet V$
is the spinor representation. (Looijenga-Lunts)
- X HK variety
$\mathfrak{g}(X;\mathbb{Q}) = \mathfrak{so}(H^2(X;\mathbb{Q}) \oplus U)$
(Looijenga-Lunts, Verbitsky) hyperbolic plane

<u>Thm</u> (Taelman) X, Y
sm proj var / $\mathbb{C}$, admitting

a holo sympl form, $\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$. Then Φ induces an isom

$$\Phi^{\mathfrak{g}} : \mathfrak{g}(X;\mathbb{Q}) \xrightarrow{\sim} \mathfrak{g}(Y;\mathbb{Q})$$

st Φ^{H} is $\Phi^{\mathfrak{g}}$-equiv

<u>Prop</u> $\mathfrak{g}(X;\mathbb{Q})$ is a semisimple Lie algebra

<u>Condition (*)</u> $\mathfrak{g}(X;\mathbb{Q})$ does not have any

simple factors of type A_1, C_2, A_3, D_4.

A_n: $2\mathbb{L}_{n+1}$ $\quad 2F_4 \quad 2E_8$

Computing $g(X)$

<u>Thm</u> (Beauville - Bogomolov)
X sm proj var $/\mathbb{C}$, ω_X trivial, then $\exists$ étale covering

$$\prod X_i \times \prod Y_j \times A \longrightarrow X$$

X_i: HK

Y_j: strict CY

A: ab var

- $H^i(\mathcal{O}_{Y_j}) = 0$ for $0 < i < \dim Y_j$
- $\dim Y_j \geq 3$

<u>Cor</u> X sympl $\Rightarrow$
$\exists X_i$ HK, ab var A_X
G finite grp, $G \curvearrowright \prod X_i \times A_X$
st $X \cong (\prod X_i \times A_X)/G$
$Y \cong (\prod Y_j \times A_Y)/H$
$\Rightarrow H^\cdot(X) = (\bigotimes H^\cdot(X_i) \otimes H^\cdot(A_X))^G$

<u>Lemma</u> $\mathfrak{g}(X) =$
$\mathfrak{g}(X)_{HK} \times (\mathfrak{so}(H^1(A_X) \oplus H^1(A_X)^\vee))^{G'}$

Lemma $\Phi^g : g(X) \xrightarrow{\sim} g(Y)$ is a product of isoms $\left(\underline{2\mathcal{O}}\left(H^1(A_x)\oplus H^1(A_x)^{\vee}\right)^{\mathcal{G}}\right) \xrightarrow[\Phi^g_{Ab}]{\sim} \left(\underline{2\mathcal{O}}\left(H^1(A_y)\oplus H^1(A_y)^{\vee}\right)^{H}\right)$, and $g(X)_{HK} \xrightarrow{\sim} g(Y)_{HK}$ (Needs $(*)$)

Def $\tilde{H}(A_x) = H^1(A_x) \oplus H^1(A_x)^{\vee}$

Strategy for main thm

1) Show Φ^g_{Ab} comes

from any "equivariant" iso $\varphi : \hat{H}(A_X) \xrightarrow{\sim} \hat{H}(A_Y)$

2) Find $\psi : \hat{H}(A_Y) \xrightarrow{\sim} \hat{H}(A_Y)$ st $\psi \circ \varphi : \hat{H}(A_X) \xrightarrow{\sim} \hat{H}(A_Y)$ respects grading and HS.

3) Do something similar for $\mathfrak{g}(X)_{HK} \to \mathfrak{g}(Y)_{HK}$

4) Lift the autom ψ to an autom Ψ of $H^{\cdot}(Y)$ st

$\Phi \circ \Phi^{H} : H^{\cdot}(X;\mathbb{Q}) \cong H^{\cdot}(Y;\mathbb{Q})$
respects grading + HS.

<u>Problem:</u> $G \neq H$?

$\rightsquigarrow$ Replace G by $\mathbb{Q}[G]$

$G \curvearrowright H^1(A_x)$

$\rightsquigarrow$ $H^1(A_x)$ is a $\mathbb{Q}[G]$-module

$\mathbb{Q}[G] = R_1 \times \cdots \times R_k$,
R_i simple $\mathbb{Q}$-algebra

Def $R_x = \prod R_i$

$R_i \hookrightarrow H^?(A_x)$ nontrivial

Thm $\exists$ isom $\rho : R_x \xrightarrow{\sim} R_y$, and $\bar{\Phi}^g_{Ab}$ is the adjoint of a ρ-equiv isom $\tilde{H}(A_x) \xrightarrow{\sim} \tilde{H}(A_y)$ (need (*))

Pf sketch for simplicity, assume $R_x = R_i$

Then $R_y = R_j \subseteq \mathbb{Q}[H]$

Wedderburn:

$R_x = M_k(D)$

D div. alg

$R_y = M_l(E)$

E div alg.

$\mathfrak{Z}(\hat{H}(A_x))^G \subseteq \operatorname{End}(\hat{H}(A_x))^G$

$\Phi^g_{Ab} \downarrow \wr \qquad \qquad \qquad \qquad \qquad \qquad M_n(D^{op})$

$\mathfrak{Z}(\hat{H}(A_y))^H \subseteq M_m(E^{op})$

Jacobson: Φ^g_{Ab} lifts to

an isom $M_n(D^{op}) \xrightarrow{\sim} M_m(E^{op})$

$M_m(E)$

$\Rightarrow D \cong E$. Also $k = l$

$\Rightarrow R_x \cong R_y$

Skolem-Noether gives

$\hat{H}(A_x) \xrightarrow{\sim} \hat{H}(A_y)$. $\square$

$\dim H^i(\mathcal{O}_Y) = \dim H^0(\Omega^i_Y)$

sympl form σ on X

$\sigma \in H^0(\Omega^2_X)$

$H^0(\Omega^2_{Y_j}) = 0$