

Derived equivalences of holomorphic symplectic varieties

Motivation

X, Y sm proj var / \mathbb{C}

Thm (Orlov) $\Phi: D^b(X) \xrightarrow{\sim} D^b(Y)$
induces an isom

$$\Phi^H: H^*(X, \mathbb{Q}) \xrightarrow{\sim} H^*(Y, \mathbb{Q})$$

• Φ^H respects the columns of the Hodge diamond.

$\Delta \Phi^H$ does not respect

the grading.

Conj If $\phi: D^b(X) \xrightarrow{\sim} D^b(Y)$
then \exists isom

$H^i(X; \mathbb{Q}) \xrightarrow{\sim} H^i(Y; \mathbb{Q})$
that respects both
the grading & HS.

Known:

- X, Y (anti)-ample
canonical bundle.
- Abelian (Mukai)
- X, Y hyperkähler
(Taelman)

$\text{Thm}(L)$ X, Y sm proj
 var / \mathbb{C} , admitting a
 holomorphic symplectic
 form and satisfying
 (*). If $D^3(X) \cong D^3(Y)$
 then \exists isom $H^i(X; \mathbb{Q}) \cong$
 $H^i(Y; \mathbb{Q})$
 respecting grading + HS.

GLV algebra

Hard Lefschetz Thm

X sm proj var / \mathbb{C}
of dim n . Then
 $\exists \omega \in H^2(X; \mathbb{C})$ st
multiplication by
 $\omega^k : H^{n-k}(X; \mathbb{C}) \rightarrow$

$H^{n+k}(X; \mathbb{C})$
is an isom $\forall k \leq n$.

Let $e_\omega : H^*(X) \rightarrow H^*(X)$
be $\alpha \mapsto \omega \cup \alpha$

Jacobson-Morozov Lemma

$\exists F_\omega: \mathfrak{H}^0(X) \rightarrow \mathfrak{H}^0(X)$
 $\exists f_\omega^k: \mathfrak{H}^{n+k}(X) \rightarrow \mathfrak{H}^{n+k}(X)$
is the inverse of e_ω^k .

Furthermore

$$\langle e_\omega, h, f_\omega \rangle \cong \mathfrak{sl}_2(\mathbb{C})$$

$$h: \mathfrak{H}^0(X) \rightarrow \mathfrak{H}^0(X)$$

$$\alpha \in \mathfrak{H}^{n+k}(X)$$

$$\Rightarrow h\alpha = k\alpha$$

Def The LLV algebra

(Looijenga-Lunts-Verbitsky Lie algebra)

is the subalgebra of $\text{End}(H^i(X; \mathbb{Q}))$ generated by all \mathcal{R}_2 -triples $\langle e_w, h, f_w \rangle$ where w satisfies the HL thm.

Notation: $\mathcal{J}(X; \mathbb{Q})$

Ex. • $X = \mathbb{P}^n$ $n \geq 1$

$$H^i(X; \mathbb{P}^n) = \begin{cases} 0 & \text{else} \\ \mathbb{Q} & i \text{ even} \\ & 0 \leq i \leq n \end{cases}$$

$$\Rightarrow \mathcal{J}(\mathbb{P}^n; \mathbb{Q}) = \mathcal{R}_2(\mathbb{Q})$$

• X ab var, $V = H^1(X; \mathbb{Q})$

$$g(X) = \underline{SO}(V \oplus V^{\vee})$$

$$g(X) \cap H^0(X) = \wedge^2 V$$

is the spinor representation.

(Levi-Schiff-Lunts)

• X HK variety

$$g(X; \mathbb{Q}) = \underline{SO}(H^2(X; \mathbb{Q}) \oplus U)$$

(Levi-Schiff-Lunts
Verbitsky)

hyperbolic
plane \rightarrow

Thm (Taelman) X, Y

sm proj var / \mathbb{C} , admitting

a holomorphic symplectic form,
 $\phi: D^b(X) \xrightarrow{\cong} D^b(Y)$. Then
 ϕ induces an isom

$\phi^{\#}: \mathfrak{g}(X; \mathbb{C}) \xrightarrow{\cong} \mathfrak{g}(Y; \mathbb{C})$
St $\phi^{\#}$ is $\phi^{\#}$ -equiv

Prop $\mathfrak{g}(X; \mathbb{C})$ is a
semisimple Lie algebra

Condition (*) $\mathfrak{g}(X; \mathbb{C})$

does not have any

Simple factors of type A_1, C_2, A_3, D_4 .

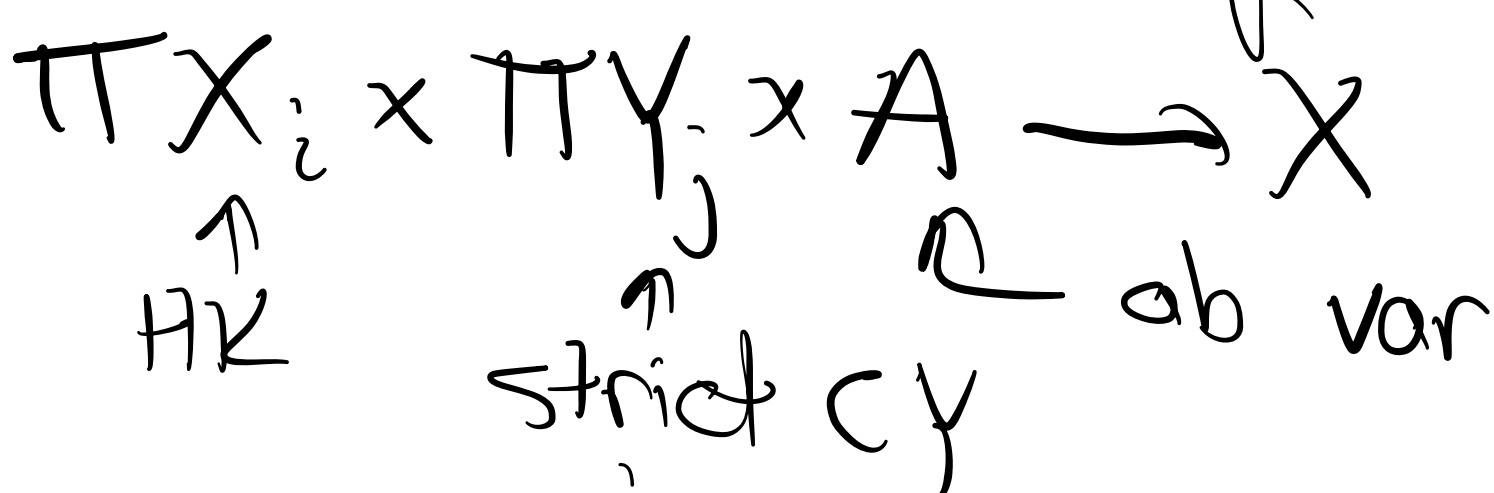
$A_n: 2b_{n+1}$ $2F_4$ $2e_8$

Computing $g(X)$

Thm (Beauville - Bogomolov)

X sm proj var / \mathbb{C} ,
 ω_X trivial, then

\exists étale covering



- $H^i(\mathbb{C}P_{Y_j}) = 0$ for
- $\dim Y_j \geq 3$, $0 < i < \dim Y_j$

For $X \text{ Sympl} \Rightarrow$

$\exists X_i \text{ HK}$, ab var A_x
 G finite grp, $G \curvearrowright \prod X_i \times A_x$

$\Rightarrow X \cong (\prod X_i \times A_x) / G$
 $Y \cong (\prod Y_j \times A_y) / H$

$\Rightarrow H^i(X) = \bigoplus H^i(X_i) \oplus H^i(A_x)$

Lemma

$H^i(X) =$

$H^i(X)_{\text{HK}}$

$X \left(\bigoplus (H^i(A_x) \oplus H^i(A_x)^{G'}) \right)$

Lemma $\mathbb{F} : g(X) \xrightarrow{\sim} g(Y)$

is a product of g

isom $(\mathbb{F} : H^1(A_x) \oplus H^1(A_x) \xrightarrow{\sim} H^1(A_y) \oplus H^1(A_y))$

and

$g(X)_{HK} \xrightarrow{\sim} g(Y)_{HK}$

(Needs $(*)$)

Def $\tilde{H}(A_x) = H^1(A_x) \oplus H^1(A_x)^V$

Strategy for main thm

1) Show $\mathbb{F} : \tilde{H}(A_x) \xrightarrow{\sim} \tilde{H}(A_y)$ comes

from an "equivariant
isom $\varphi: \hat{H}(A_x) \xrightarrow{\sim} \hat{H}(A_y)$

2) Find $\psi: \hat{H}(A_y) \xrightarrow{\sim} \hat{H}(A_y)$

st $\psi \circ \varphi: \hat{H}(A_x) \xrightarrow{\sim} \hat{H}(A_y)$
respects grading and
HS.

3) Do something similar
for $\mathcal{G}(X) \xrightarrow{HK} \mathcal{G}(Y)$

4) Lift the autom
 ψ to an autom
 Ψ of $H(Y)$ st

$\mathbb{F} \circ \mathbb{F}^H : H^1(X; \mathbb{Q}) \rightarrow H^1(X; \mathbb{Q})$
respects grading + HS.

Problem: $G \neq H$?

\leadsto Replace G by
 $\mathbb{Q}[G]$

$G \cap H^1(A_x)$

$\leadsto H^1(A_x)$ is a $\mathbb{Q}[G]$ -
module

$\mathbb{Q}[G] = R_1 \times \dots \times R_k,$

R_i simple \mathbb{Q} -algebra

Def $R_x = \prod R_i$
 $R_i \cap H^1(A_x)$
nontrivial

Thm \exists isom $p: R_x \xrightarrow{\sim} R_y$
and Φ_{AB}^{\pm} is the adjoint
of a p -equiv isom
 $H^2(A_x) \xrightarrow{\sim} H^2(A_y)$
(need $(*)$)

Pf sketch for

Simplicity, assume $R_x = R_i$
Then $R_y = R_j \subseteq \mathcal{O}[H]$

Wedderburn:

$$R_x = M_k(D)$$

\uparrow div. alg

$$R_y = M_l(E)$$

\uparrow div alg.

$$\underline{R_x}(\hat{H}(A_x)) \subseteq \text{End}_G(\hat{H}(A_x))$$

$$\begin{array}{c} \Phi \\ \downarrow \\ A \end{array}$$

$$M_n(D^{\text{op}})$$

$$\underline{R_y}(\hat{H}(A_y))^H$$

$$\subseteq M_m(F^{\text{op}})$$

Jacobson: Φ lifts to

an isom

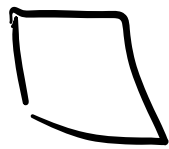
$$M_n(D^{\text{op}}) \cong \dots$$

$\Rightarrow D \cong E$. Also $k = l$ $N_m(E)$

$$\Rightarrow R_x \cong R_y$$

Skolem-Noether gives

$$\hat{H}(A_x) \xrightarrow{\cong} \hat{H}(A_y)$$



$$\dim H^i(\mathcal{O}_y) = \dim H^0(\Omega_y^i)$$

symp form σ on X

$$\sigma \in H^0(\Omega_x^2)$$

$$H^0(\Omega_{y_j}^2) = 0$$