Aspects of The Derived Torelli Theorem for K3 Surfaces

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§ Introduction

We work with smooth, projective varieties $/\mathbb{C}$.

Notation: $\mathcal{D}^b(X) = \mathcal{D}^b(\mathrm{Coh}(X))$

General question: What geometric data does $\mathcal{D}^b(X)$ "know" about X?

Examples:
- $\dim X$ (i.e. $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \Rightarrow \dim Y = \dim X$.)
- $b_1(X) := \mathrm{rk}\, H^1(X, \mathbb{Z})$
- $HH(X)$

Theorem [Orlov]. If $\pm K_X$ is ample, then $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \iff X \cong Y$.

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If A is an Abelian variety, then $\mathcal{D}^b(A) \cong \mathcal{D}^b(A^\vee)$ even when $A \ncong A^\vee$.

(Recall: $K_A \cong \mathcal{O}_A$).

Today: We will study $\mathcal{D}$-equivalence for certain K-trivial varieties:

Def. A K3 surface is a smooth, (projective), 2-dimensional variety X s.t.

1) $K_X \cong \mathcal{O}_X$
2) $H^1(X, \mathcal{O}_X) = 0$. $(\Rightarrow \pi_1(X) = 0)$.

[!] K3 surfaces "usually" have non-trivial Fourier–Mukai partners.

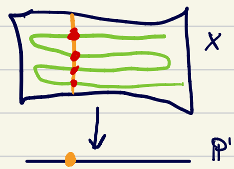
For example:

§ First examples of $\mathcal{D}$-equivalent K3 surfaces

<u>Set-up</u>: $\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ an elliptic K3 surface (i.e. $\begin{array}{c} X_\eta \\ \downarrow \\ \mathrm{Spec}(k(\eta)) \end{array}$ is a smooth curve of $g=1$)

For $k \in \mathbb{Z}$, we consider $\begin{array}{c} J^k X_\eta \\ \downarrow \\ \mathrm{Spec}(k(\eta)) \end{array}$:= the smooth curve parametrising line bundles of degree k on X_η.

There is a unique relatively minimal surface $\begin{array}{c} J^k X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ s.t. $(J^k X)_\eta \cong J^k X_\eta$.



$\gcd(k, t) = 1$

multisection index

[!] $J^k X$ is a K3 surface

<u>Theorem</u> [Mukai, Bridgeland, Căldăraru]. $\mathcal{D}^b(J^k X) \cong \mathcal{D}^b(X)$ (sometimes).

§ The Derived Torelli Theorem

$\Phi^{\mathcal{E}^\bullet}(k(x)) \cong \mathcal{E}^\bullet_x \in \mathcal{D}^b(Y)$.

so x "parametrises" $\mathcal{E}^\bullet_x$.

image of skyscrapers

Fourier–Mukai partnership

$\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$

$\Phi^{\mathcal{E}} : \mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$

$\Rightarrow \Phi_H^{v(\mathcal{E})} : H^\bullet(X, \mathbb{Q}) \cong H^\bullet(Y, \mathbb{Q})$

?

universal object

Moduli spaces of (complexes of) sheaves

?

?

Hodge structures on (rational) cohomology

$H^\bullet(X, \mathbb{Q}) \cong H^\bullet(Y, \mathbb{Q})$

If $\mathcal{E}^\bullet \in \mathcal{D}^b(X \times Y)$ universal sheaf, then $\Phi^{\mathcal{E}^\bullet} : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(Y)$ might be an equivalence

Theorem. [Derived Torelli Theorem, [Mukai, Orlov]]

Let X, Y be K3 surfaces. TFAE:

(i) $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$

(ii) $\widetilde{H}(X,\mathbb{Z}) \cong \widetilde{H}(Y,\mathbb{Z})$ (Hodge isometry).

(iii) $Y \cong M(v)$ for some $v \in N(X)$ (well-chosen)

Need to understand:

* Hodge isometries
* Moduli of sheaves.

§ Moduli spaces of sheaves on K3 surfaces

<u>Recall</u>: If $\mathcal{E}^\bullet \in \mathcal{D}^b(X)$, the Mukai vector of $\mathcal{E}^\bullet$ is

$$v(\mathcal{E}^\bullet) := ch(\mathcal{E}) \cdot \sqrt{td_X} \in H^*(X, \mathbb{Q}).$$

For a K3 surface: $v(\mathcal{E}^\bullet) \in N(X) := H^0(X,\mathbb{Z}) \oplus H^{1,1}(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$

$$\cong \mathbb{Z} \oplus NS(X) \oplus \mathbb{Z}$$

And: $\sqrt{td_X} = (1, 0, 1)$.

<u>Examples</u>:

- $x \in X$ closed : $v(k(x)) = (0, 0, 1)$
- $\mathcal{O}_X(D) \in \mathrm{Pic}(X)$: $v(\mathcal{O}_X(D)) = (1, D, \frac{1}{2}D^2)$
- $\mathcal{F} \in \mathrm{Coh}(X)$: $v(\mathcal{F}) = (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})))$

Def. [Mukai] For $v \in N(X)$ (effective), we denote

$$\mathcal{M}(v) := \{ \mathcal{F} \in Coh(X) \mid v(\mathcal{F}) = v \}$$

([!] I'm ignoring something important : stability)

Theorem. [Mukai, Göttsche - Huybrechts, O'Grady, Yoshioka]

If v effective, primitive (i.e. $\frac{v}{n} \in N(X) \Rightarrow n = \pm 1$). Then $\mathcal{M}(v)$ is a smooth, projective hyperkähler variety that is deformation equivalent to a Hilbert scheme of points on a K3 surface. Moreover,

$$\dim \mathcal{M}(v) = v^2 + 2$$

Hence $\mathcal{M}(v)$ is a K3 surface if $v^2 = 0$.

p.s. $N(X)$ has an integral bilinear form $N(X) \times N(X) \longrightarrow \mathbb{Z}$

given by $\begin{cases} (0, D, 0) \cdot (0, E, 0) = D \cdot E \\ (r, 0, s) \cdot (r', 0, s') = -rs' - r's . \end{cases}$ (v^2 means $v \cdot v$).

§ Example

Recall: for $\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ elliptic K3, we have $\begin{array}{c} J^k X \\ \downarrow \\ \mathbb{P}^1 \end{array}$ parametrising line bundles supported on the fibres & having degree k there.

$\boxed{!}$ We have $J^k X \cong M(0, F, k)$

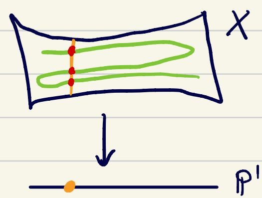
class of the fibre of $\begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array}$

Q. For which v is $M(v)$ a fine moduli space?

(Important because $M(v)$ is fine $\iff \exists$ universal sheaf $\mathcal{F} \in \mathrm{Coh}(X \times M(v))$.)

A. [Mukai, Căldăraru, Mattei-M]

$$M(v) \text{ is fine} \iff \mathrm{div}(v) := \gcd_{x \in N(X)} (v \cdot x) = 1.$$



$\mathrm{div}(0, F, k) = \gcd(k, \mathrm{div}\, F) = \gcd(k, t)$

$\hookrightarrow J^k X$ fine $\iff \gcd = 1$

§ Hodge isometries.

Recall: $H^2(X,\mathbb{Z})$ has a bilinear form $\cup : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$

and we can extend it to $H^*(X,\mathbb{Z}) \cong H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$

by $(1,0,0)\cdot(0,0,1) = -1$.

And: $H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$

both 1-dimensional

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Def. A Hodge isometry is a group homomorphism $H^2(X,\mathbb{Z}) \xrightarrow{f} H^2(X,\mathbb{Z})$ that respects both the bilinear form and the Hodge structure.

i.e.
$$\begin{cases} f(x) \cup f(y) = x \cup y \\ f(H^{2,0}(X)) = H^{2,0}(X) \end{cases}$$

Theorem (Torelli Theorem for K3 surfaces). $X \cong Y \iff H^2(X,\mathbb{Z}) \cong H^2(Y,\mathbb{Z})$

(and DERIVED Torelli says $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \iff H^*(X,\mathbb{Z}) \cong H^*(Y,\mathbb{Z})$)

Hence we can "construct" a K3 surface by specifying a "suitable" Hodge lattice.

For example. If $v \in N(X)$ effective, primitive, what is $H^2(M(v),\mathbb{Z})$?

Theorem [Mukai]. Let $\mathcal{F} \in \mathrm{Coh}(X \times M(v))$ be a (quasi-)universal sheaf. Then the cohomological FM transform

$$\Phi_H^{v(\mathcal{F})} : H^*(X,\mathbb{Q}) \longrightarrow H^*(M(v),\mathbb{Q})$$

induces a Hodge isometry $v^\perp/\mathbb{Z}\cdot v \cong H^2(M(v),\mathbb{Z})$.

§ Proof of the Derived Torelli Theorem

1. $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \Rightarrow \tilde{H}(X,\mathbb{Z}) \cong \tilde{H}(Y,\mathbb{Z})$

Is because $\Phi_H^{v(\mathcal{F})}$ maps integral classes to integral classes (computation)

2. $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \Rightarrow Y \cong M(v)$.

Let $f: \tilde{H}(X,\mathbb{Z}) \cong \tilde{H}(Y,\mathbb{Z})$ be a Hodge isometry. Let $v = f^{-1}(0,0,1)$

Then

$$\begin{array}{ccc} v & \longmapsto & (0,0,1) \\ \tilde{H}(X,\mathbb{Z}) & \cong & \tilde{H}(Y,\mathbb{Z}) \end{array}$$

$$v^\perp \cong (0,0,1)^\perp \cong H^2(Y,\mathbb{Z}) \oplus \mathbb{Z}(0,0,1)$$

$$H^2(M(v),\mathbb{Z}) \cong v^\perp/\mathbb{Z}v \cong (0,0,1)^\perp/\mathbb{Z}\cdot(0,0,1) \cong H^2(Y,\mathbb{Z})$$

3. $Y \cong M(v) \Rightarrow \mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$

The universal sheaf is the kernel of a FM equivalence.

Conclusion. The three aspects are all equally important.

FM partners of K3 surfaces

universal sheaf

cohomological FM transforms are integral

Fine moduli spaces of sheaves

Torelli & $H^2(M(v),\mathbb{Z}) \cong v^{\perp}/v$

Hodge isometries of integral Hodge structures

§ But what about the non-fine moduli spaces?

Recall: $\mathcal{M}(v)$ is fine $\iff$ $\operatorname{div} v = \gcd_w (v.w) = 1$

More generally:

$$\begin{tikzcd} T(X) \arrow[r, hook] \arrow[dr, bend right] & v^\perp \arrow[r, hook] \arrow[d] & \tilde{H}(X,\mathbb{Z}) \\ & v^\perp/v \cong H^2(\mathcal{M}(v),\mathbb{Z}) & \end{tikzcd} \qquad (T(X) := N(X)^\perp \subset \tilde{H}(X,\mathbb{Z}))$$

$$0 \longrightarrow T(X) \hookrightarrow T(\mathcal{M}(v)) \xrightarrow{\alpha} \mathbb{Z}/\operatorname{div}(v)\cdot\mathbb{Z} \longrightarrow 0$$

[!] For a K3 surface X, we have $\operatorname{Br}(X) \cong \operatorname{Hom}(T(X), \mathbb{Q}/\mathbb{Z})$

So $\alpha \in \operatorname{Br}(\mathcal{M}(v))$

Def. α is the obstruction to the existence of a universal sheaf on $X \times \mathcal{M}(v)$.

Thm. We have $\mathcal{D}^b(X) \cong \mathcal{D}^b(\mathcal{M}(v), \alpha)$

Cor. $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y) \iff T(X) \cong T(Y)$.

§ Application: Ogg – Shafarevich Theory

If $\begin{matrix} S \\ \circlearrowleft\downarrow \\ \mathbb{P}^1 \end{matrix}$ elliptic K3 with a section, then

$$\begin{array}{ccc} Ш(S) & \cong & \mathrm{Br}(S) \\ \| & & \wr\| \\ \left\{ \begin{matrix} X \\ \downarrow \\ \mathbb{P}^1 \end{matrix} \;\middle|\; J^0(X) \cong S \right\} & & \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \\ \begin{matrix} X \\ \downarrow \\ \mathbb{P}^1 \end{matrix} & \longmapsto & \alpha_X \end{array}$$

$\leftarrow$ obstruction class

And therefore $\ker\left(\alpha_X : T(S) \longrightarrow \mathbb{Q}/\mathbb{Z}\right) \cong T(X)$

$\boxed{!}$ $J^0 J^k X \cong J^0 X$, so $[J^k X] \in Ш(S)$

Thm [Căldăraru].
$$\begin{array}{ccc} Ш(S) & \xrightarrow{\sim} & \mathrm{Br}(S) \\ X & \longmapsto & \alpha_X \\ J^k X & \longmapsto & k \cdot \alpha_X . \end{array}$$

Cor. Since $\alpha_{J^k X} = k \cdot \alpha_X$, we have

$$\mathcal{D}^b(X) \simeq \mathcal{D}^b(J^k X) \iff T(X) \simeq T(J^k X) \iff \ker \alpha_X = \ker k \cdot \alpha_X$$

$$\iff \gcd(k, |\alpha_X|) = 1.$$

Now: what is $|\alpha_X|$?

Recall: $0 \longrightarrow T(X) \longrightarrow T(J^0 X) \xrightarrow{\alpha_X} \mathbb{Z}/\mathrm{div}(v)\mathbb{Z} \longrightarrow 0$

where $v = (0, F, 0)$. Thus: $\mathrm{div}(v) = \mathrm{div}\, F = t$

$$\mathcal{D}^b(X) \cong \mathcal{D}^b(J^k(X)) \iff \gcd(k, t) = 1.$$

Q: Are all FM partners of X isomorphic to some $J^k X$?
A. [M – Shinder]. No.