Derived autoequivalances on algebraic Flops.

Preamble:

DW1: - defines a new "invariant" the noncommutative deformation algebra called the contraction algebra A_{con} associated to any contractible rational curve in any 3-fold.

- Exploits this invariant to give the first intrinsic description of a derived autoequivalence associated to a general flopping curve.

August: - For any $f: U \longrightarrow \operatorname{Spec} R$, where $\operatorname{Spec} R$ is a complete local isolated cDv singularity, it's proven that all the associated contraction algebras are derived equivalent $\rightsquigarrow$ the corresponding derived category is an invariant of the singularity $\operatorname{Spec} R$.

DW3: In the setting of August above, two combinatorial objects: a finite hyperplane arrangement $\mathcal{H}$ and an infinite hyperplane arrangement $\mathcal{H}^{aff}$ can be associated to the flopping contraction $f: U \to \operatorname{Spec} R$

Definitions:

1. A flop is a commutative diagram

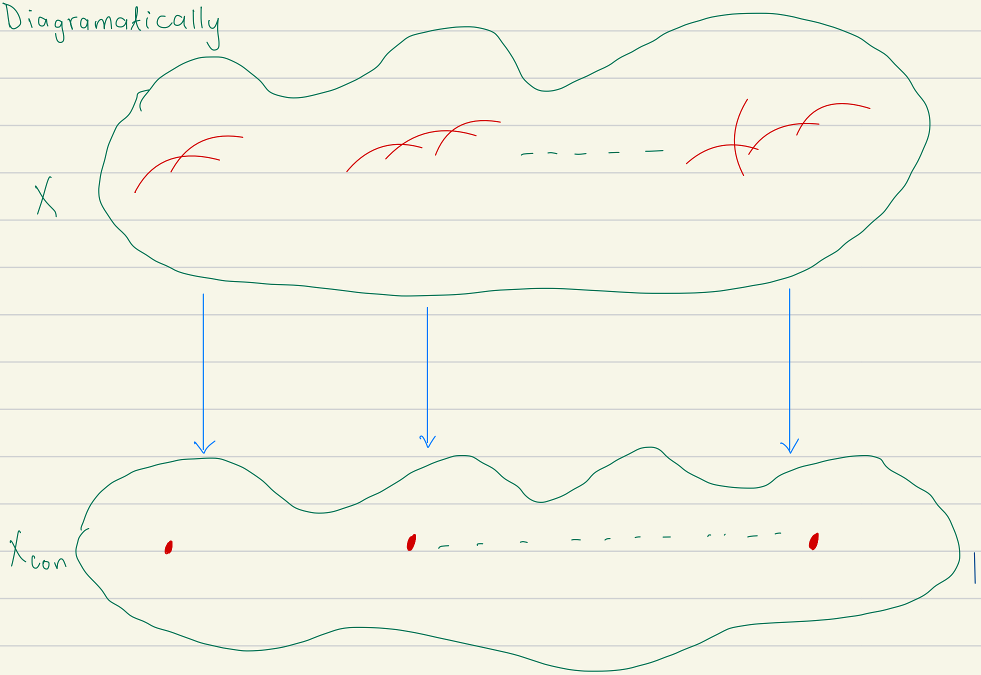
$$\begin{tikzcd}
X^- \arrow[rr, dashed, "g"] \arrow[dr, "f^-"'] & & X^+ \arrow[dl, "f^+"] \\
& Y &
\end{tikzcd}$$

where are small projective birational morphisms, and the canonical bundles are trivial over Y.

2. We will refer $f^{\pm}$ as flopping contractions.

Goal: Let $X \longrightarrow X_{con}$ be a global 3-fold flopping contraction, where X is quasi projective and has only mild singularities:

Diagramatically

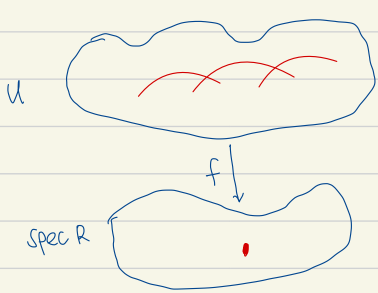


Use this to define autoequivalences $D^b(\mathrm{Coh}\, X) \longrightarrow D^b(\mathrm{Coh}\, X)$.

We work through the following set ups, then globalise these to the global setup.

1. Zariski local set up:

Suppose that $f: U \longrightarrow \operatorname{spec} R$ an algebraic flopping contraction which is an isomorphism away from precisely one point $m \in \operatorname{Max} R$. Also assume U has mild singularities.



where $U \subset X$, U has mild singularities.

By [VdB], it is well known that there exists a tilting bundle $\mathcal{V} = \mathcal{O}_U \oplus \mathcal{N}$ on U that induces a derived equivalence

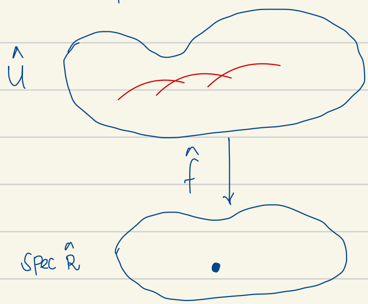
$$D^b(\operatorname{Coh} U) \xrightarrow{\mathrm{RHom}(\mathcal{V}, -)} D^b(\operatorname{mod} \Lambda), \text{ where}$$

$$\Lambda = \operatorname{End}_U(\mathcal{V}) = \operatorname{End}_R(f_* \mathcal{V}) = \operatorname{End}_R(R \oplus f_* \mathcal{N})$$

Costs of the algebraic flopping contraction.

1. There is lack of Krull schmidt since R is not complete local.
2. There are no obvious algebraic objects which correspond to the Chambers of $\mathcal{H}$ and $\mathcal{H}^{\text{aff}}$, so no obvious way of producing autoequivalences via wall crossing.

2. Complete local setup:



* From the Zariski local set up $\Lambda = \operatorname{End}_R(R \oplus f_* \mathcal{N})$, after completing $\hat{\Lambda} = \operatorname{End}_{\hat{R}}(\hat{R} \oplus \widehat{f_* \mathcal{N}})$. Set $A = \operatorname{End}_{\hat{R}}(N)$ where $N = \hat{R} \oplus \widehat{f_* \mathcal{N}}$.

* By eg. [DW3] A is basic morita equivalent to $\hat{\Lambda}$

* From preamble, we can associate $\mathcal{H}$ or $\mathcal{H}^{\text{aff}}$ to $\hat{f} : \hat{U} \longrightarrow \operatorname{Spec} \hat{R}$

* Let $\alpha : C_+ \to D$ be an atom from a choosen Chamber C_+ to another Chamber D in $\mathcal{H}$ or $\mathcal{H}^{\text{aff}}$. For some wall i of D there is a monodromy around i.

* We can associate an algebra $B := \mathrm{End}_{\hat{R}}(N_D)$ to the chamber D where N_D is some module obtained by crossing the wall i. We write $\Phi_\alpha : D^b(\mathrm{End}_{\hat{R}}(N)) \longrightarrow D^b(\mathrm{End}_{\hat{R}}(N_D))$ for the associated derived equivalence (also known as mutation)

* B inherits $n+1$ idompotents e_i corresponding to the walls of D, thus there is a ring homomorphism $B \longrightarrow B_i$ with $B_i = B/_{1-e_i}$

* For $f : U \longrightarrow \mathrm{Spec}\, R$, to construct an autoequivalence on U associated to the above monodromy. We consider Λ, that is derived equivalent to u and construct the functor diagram.

$$\psi_{\alpha,i} : \begin{tikzcd} D(B_i) \arrow[r, "H"] & D(B) \arrow[r, "\Phi_\alpha"] & D(A) \arrow[r, "\mathbb{F}"] & D(\hat{\Lambda}) \arrow[r, "F"] & D(\Lambda) \arrow[llll, bend right, "\psi^{LA}_{\alpha,i}"'] \arrow[llll, bend left, "\psi^{RA}_{\alpha,i}"] \end{tikzcd}$$

Proposition: For any choice of (α, i), there excists ${}_\Lambda C_\Lambda = C_{\alpha,i}$ in the derived category of Λ-Λ bimodules such that given the functorial diagram

$$\begin{tikzcd} D(B_i) \arrow[r, "\psi_{\alpha,i}"] & D(\Lambda) \arrow[l, bend right, "\psi^{LA}_{\alpha,i}"'] \arrow[l, bend left, "\psi^{RA}_{\alpha,i}"] \end{tikzcd}$$

and setting $\mathrm{Twist}_{\alpha_i} := \mathrm{RHom}_\Lambda(C_\Lambda, -)$ and $\mathrm{Twist}^*_{\alpha,i} := - \otimes^L_\Lambda C_\Lambda$, there are functorial triangles.

1. $\psi_{\alpha,i} \circ \psi^{RA}_{\alpha,i} \longrightarrow \mathrm{Id}_\Lambda \longrightarrow \mathrm{Twist}_{\alpha,i} \longrightarrow$
2. $\mathrm{Twist}^*_{\alpha,i} \longrightarrow \mathrm{Id}_\Lambda \longrightarrow \psi_{\alpha,i} \circ \psi^{LA}_{\alpha,i} \longrightarrow$

Theorem: For any choice of (α, i), $C_{\alpha,i}$ is a two-sided tilting complex, giving rise to the autoequivalence $\mathrm{Twist}_{\alpha,i}$ of $D^b(\mathrm{mod}\,\Lambda)$.

* Let $f: U \to \operatorname{Spec} R$ be an algebraic flopping contraction, then we know:

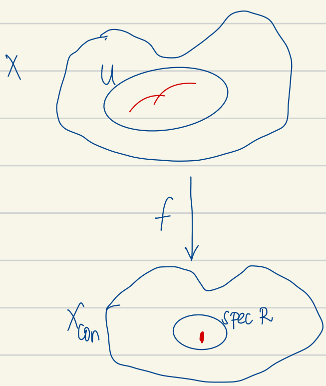
$$\begin{tikzcd}
D^b(\mathrm{coh}\, U) \arrow[r, "\Omega := \mathbf{R}\mathrm{Hom}_U(\mathcal{V}, -)"] & D^b(\mathrm{mod}\,\Lambda) \arrow[l, bend right=40, "\Omega^{LA}"'] \arrow[l, bend left=40, "\Omega^{RA}"]
\end{tikzcd}$$

We define a functor $\mathrm{GeoTwist}_{\alpha,i}$:

$$\begin{tikzcd}
D^b(\mathrm{coh}\, U) \arrow[r, "\mathbf{R}\mathrm{Hom}(\mathcal{V}, -)"] \arrow[d, dashed] & D^b(\mathrm{mod}\,\Lambda) \arrow[d, "\mathrm{Twist}_{\alpha,i}"] \\
D^b(\mathrm{coh}\, U) & D^b(\mathrm{mod}\,\Lambda) \arrow[l, "\otimes^{L}_{\Lambda} \mathcal{V}"']
\end{tikzcd}$$

* $\mathrm{GeoTwist}_{\alpha,i}$ is composition of equivalences.

* For the global flopping contraction $X \to X_{con}$, to construct an autoequivalence, choose an affine open subset $\operatorname{Spec} R$ of X_{con} containing only one point P_k and set U to be its preimage such that.



X_{con} is covered by finitely affine open subsets, we combine the auto equivalences on U_i to give an autoequivalence on X.

$$D^b(\mathrm{Qcon}\, X) \longrightarrow D^b(\mathrm{Qcon}\, X).$$

* There is a group homomorphism $PBr \longrightarrow \mathrm{Autoeqiv}(D^b(\mathrm{Coh}X))$

$\not\downarrow$

there are lots of autoequivs on $D^b(\mathrm{Coh}X)$