Categorical Torelli theorems for Fano threefolds

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Fano threefolds

Let X be a smooth Fano threefold with $\operatorname{Pic} X = \mathbf{Z} \cdot H$. These are classified by their

- *index*: $K_X = -\iota_X H$, and
- *degree*: $d = H^3$.

$\mathbf{K_X = -H}$		$\mathbf{K_X = -2H}$	
d	**SOD**	**d**	**SOD**
22	...	5	...
18	...	4	...
16	...		
14	...	3	$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$
12	...		
10	$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$	2	...
8	...		
6	...	1	...
4	...		
2	$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X \rangle$		

Theorem (Bondal–Orlov)

$\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \implies X \simeq X'$.

Motivation

Example

Let $X \subset \mathbf{P}^4$ be a cubic threefold. Then $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$ where

$$\mathcal{A}_X = \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}^\bullet(\mathcal{O}_X, E) = \mathrm{Hom}^\bullet(\mathcal{O}_X(1), E) = 0\}$$

is the *Kuznetsov component*.

Natural question

Does $\mathcal{A}_X$ determine X up to isomorphism?

Motivation

For cubic threefolds, the answer is **yes**.

Theorem (Bernardara–Mehrotra–Macrì–Stellari)

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies X \simeq X'$.

Proof idea.

- Consider moduli spaces of Bridgeland stable objects in $\mathcal{A}_X$. We get $M_\sigma(\mathcal{A}_X, [I_\ell]) \simeq \mathrm{Hilb}_\ell(X)$.
- An equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ induces an isomorphism of the corresponding moduli spaces, hence an isomorphism of the Hilbert schemes of lines.
- $J(X) = \mathrm{Alb}(\mathrm{Hilb}_\ell(X))$
- Hence $J(X) \simeq J(X')$
- Hence $X \simeq X'$ by the classical Torelli theorem for cubic threefolds due to [Clemens–Griffiths, Tyurin].

□

Question

What about the other Fano threefolds?

Gushel–Mukai (GM) threefolds

Let X be a Fano threefold with $\operatorname{Pic} X = \mathbf{Z} \cdot H$, $d = H^3 = 10$, and $K_X = -H$. Then

$$X = \mathsf{Gr}(2,5) \cap (\text{quadric}) \cap \mathbf{P}^7$$

or

$$X = \text{double cover of } \mathsf{Gr}(2,5) \cap \mathbf{P}^6 \text{ branched in a quadric hypersurface}$$

and it has an SOD $\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}^\vee \rangle$ where $\mathcal{E}$ is the restriction of the tautological bundle on $\mathsf{Gr}(2,5)$.

Question

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies ??$

Moduli of stable objects in $\mathcal{A}_X$

Lots is known classically about *conics* $C \subset X$ on GM threefolds, so let's consider $M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)])$.

Lemma (J.–Lin–Liu–Zhang)

$M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)]) \simeq$ *minimal surface of* $\mathrm{Hilb}_C(X) =: \mathrm{mHilb}_C(X)$.

Proof idea.

- Certain $I_C \in \mathrm{Hilb}_C(X)$ are s.t. $I_C \in \mathcal{A}_X$. Others are s.t. $I_C \notin \mathcal{A}_X$.
- In all cases, $\mathrm{pr}(I_C) \in \mathcal{A}_X$ are Bridgeland stable.
- There is a $L \simeq \mathbf{P}^1$ of conics in $\mathrm{Hilb}_C(X)$ s.t. $I_C \notin \mathcal{A}_X$.
- $\mathrm{pr}\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X$ induces a map $p\colon \mathrm{Hilb}_C(X) \to M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)])$.
- L is contracted to a point by p.
- It follows that $M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)]) \simeq \mathrm{mHilb}_C(X)$.

□

Birational categorical Torelli for GM threefolds

Theorem (JLLZ)

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \implies X \sim_{\mathrm{bir}} X'$.

Proof idea.

- Suppose $\Phi\colon \mathcal{A}_X \to \mathcal{A}_{X'}$ is an equivalence.
- Where does Φ map $M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)])$?
- For numerical reasons, either
  1. $\Phi(M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)])) \simeq M_\sigma(\mathcal{A}_{X'}, [\mathrm{pr}'(I_{C'})])$ or
  2. $\Phi(M_\sigma(\mathcal{A}_X, [\mathrm{pr}(I_C)])) \simeq M_\sigma(\mathcal{A}_{X'},$ certain other class)
- Hence,
  1. $\mathrm{mHilb}_C(X) \simeq \mathrm{mHilb}_C(X')$ or
  2. $\mathrm{mHilb}_C(X) \simeq \mathrm{mHilb}_C($certain birational transformation of $X')$
- Classical results due to [Debarre–Iliev–Manivel] + [Logachev] give $X \sim_{\mathrm{bir}} X'$ in both cases.

□

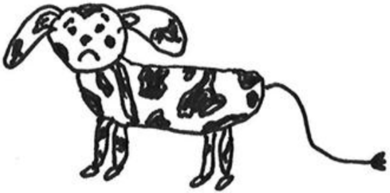
Another Fano threefold

Now let X be a Fano threefold with $\operatorname{Pic} X = \mathbf{Z} \cdot H$, $K_X = -H$, and $d = H^3 = 2$. These are double covers of $\mathbf{P}^3$ branched in a sextic hypersurface Z. They have SODs $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X \rangle$.

Question: Can we do the same thing as before?

Remark

Due to a technical issue re: stability conditions on $\mathcal{A}_X$ in this case, it's not clear whether the previous method is possible.



An equivariant approach

Recall that $X \to \mathbf{P}^3$ is a double cover branched in a sextic hypersurface, with $\mathrm{D}^\mathrm{b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X \rangle$. So there's an action of $\mu_2 = \{1, -1\}$ on X.

Consider the *equivariant derived category* of X w.r.t this action:

$$\mathrm{D}^\mathrm{b}([X/\mu_2]) \simeq \mathrm{D}^\mathrm{b}(X)^{\mu_2} = \langle \mathcal{A}_X^{\mu_2}, \mathcal{O}_X^{\mu_2} \rangle.$$

There is a theorem [Collins–Polishchuk, Ishii–Ueda, Kuznetsov–Perry] which gives *another* SOD for such covers, in terms of the *base* and *branch locus* Z.

Shuffling the two SODs around gives

Lemma (Dell–J.–Rota)

$\mathcal{A}_X^{\mu_2} = \langle \mathrm{D}^\mathrm{b}(Z), \textit{exceptional} \rangle.$

Another categorical Torelli theorem

Theorem (DJR, Lin–Zhang)

Let X be very general and $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$ be Fourier–Mukai. Then $X \simeq X'$.

Proof idea.

- Begin with an equivalence $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$.
- This descends (!) to an equivariant equivalence $\Phi^{\mu_2}\colon \mathcal{A}_X^{\mu_2} \simeq \mathcal{A}_{X'}^{\mu_2}$.
- Consider the *topological K-theory* $K_0^{\mathrm{top}}(\mathcal{A}_X^{\mu_2})$. One can put a Hodge structure on this. Furthermore, one can show $K_0^{\mathrm{alg}}(\mathcal{A}_X^{\mu_2})^{\perp} \simeq H^2_{\mathrm{prim}}(Z, \mathbf{Q})$ (which preserves the Hodge structure).
- Thus Φ^{μ_2} induces a Hodge isometry $H^2_{\mathrm{prim}}(Z, \mathbf{Q}) \simeq H^2_{\mathrm{prim}}(Z', \mathbf{Q})$.
- Classical Torelli [Donagi] $\implies Z \simeq Z' \implies X \simeq X'$.

□

Thank you!