Tilting sheaves
and a Baer-Bondal theorem

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Daniel Bernal

Universidade Estadual de Campinas

Advisor: Ph.D. Marcos Benevenuto Jardim

Co-advisor: Ph.D. Cristian Mauricio Martínez Esparza

Notation

- $\mathcal{A}$ abelian category,
- $D^b(X) := D^b(\mathrm{Coh}(X))$,
- all vector spaces over $\mathbb{C}$,
- RF and LG are the right and left derived functors when they exist,
- $\mathrm{Mod}(A)$ abelian category of $A-$modules
- $\mathrm{mod}(A)$ abelian category of fg $A-$modules,
- X smooth projective over $\mathbb{C}$

The category of representations of quivers

Two important quivers

Kronecker quiver

$$\bullet \rightrightarrows \bullet$$

Beilinson quiver for $\mathbb{P}^n$

Representation of a quiver

- A representation of a quiver is just a functor $\mathcal{C}Q \longrightarrow \mathrm{Vec}_{\mathbb{C}}$.
- A morphism is just a natural transformation.

Example

The representations of $\bullet \rightrightarrows \bullet$ of dimension $(1, 1)$ are

$$\mathbb{C} \underset{w_2}{\overset{w_1}{\rightrightarrows}} \mathbb{C}$$

Isoclasses of representations are parameterised by
$$\mathbb{C}^2 - \{0\} \setminus \mathbb{C}^* = \mathbb{P}^1$$

Path algebra

We assume Q connected.

$\mathbb{C}-$*algebra* $\mathbb{C}Q$

- basis is the set of paths,
- product is the concatenation of paths

$\mathbb{C}Q$ is graded and associative:

$$\mathbb{C}Q = \bigoplus_{k \geqslant 0} (\mathbb{C}Q)_k; \quad (\mathbb{C}Q)_0 = \{e_i\}_{i \in Q_0}$$

Such e_i are orthogonal idempotents, and $1 = \sum e_i$.

$$\dim_{\mathbb{C}} \mathbb{C}Q < +\infty \iff Q \text{ has no oriented cycles}$$

A neat equivalence

Let A a $\mathbb{C}$—algebra, A^{op} is the opposite algebra

$$a \cdot b = ba$$

- $\mathrm{mod}(A^{\mathrm{op}})$ category of f.g. right A—modules,
- $\mathbb{C}Q^{\mathrm{op}} = (\mathbb{C}Q)^{\mathrm{op}}$

Proposition

$$\mathrm{rep}_{\mathbb{C}}(Q) \cong \mathrm{mod}(\mathbb{C}Q)$$

Proof and abelianity(?)

$$W = (W_i, w_a) \rightsquigarrow M = \bigoplus_{i \in Q_0} W_i$$

$$M \in \mathrm{mod}(\mathbb{C}Q) \rightsquigarrow \begin{cases} W_i = e_i M \\ w_a : W_{s(a)} \longrightarrow W_{t(a)} \, \forall a \in Q_1 \end{cases}$$

This instantly shows that $\mathrm{rep}_{\mathbb{C}}(Q)$, $\mathrm{rep}_{\mathbb{C}}(Q^{\mathrm{op}})$ are abelian

Relations on a quiver

A **relation** on Q is a $\mathbb{C}-$linear combination of paths of length $\geqslant 2$, with the same head and tail.

$$\text{finite set of relations} \longleftrightarrow \text{two-sided ideal } \langle R \rangle$$

We can consider representations of (Q, R), R being a set of relations.

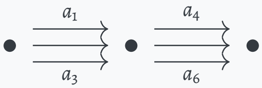
$$\text{rep}_{\mathbb{C}}(Q, R) \text{ category of f.d. representations of } (Q, R)$$

Proposition

$$\text{rep}_{\mathbb{C}}(Q, R) \cong \text{mod}\left({}^{\mathbb{C}Q}\!\big/_{\langle R \rangle} \right)$$

An example

Beilinson for $\mathbb{P}^2$



$$R = \{a_1a_5 - a_2a_4,\ a_1a_6 - a_3a_4,\ a_2a_6 - a_3a_5\}$$

This relations correspond to the commutativity of the linear forms x, y, z in $\mathbb{P}^2$.

$$\mathbb{C}Q/_{\langle R\rangle} \cong \operatorname{End}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\right)$$

Simple objects and JH-filtrations

$S(i)$ for all $i \in Q_0$ are a set of simple objects in $\mathrm{rep}_{\mathbb{C}}(Q, R)$. If A, B are simple objects in $\mathcal{A}$,

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

Jordan-Hölder filtration of $E \in \mathcal{A}$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that $F_i = {E_i}/{E_{i-1}}$ is simple. Such factors are **unique!**

E can be obtained by extensions of simple objects.

Finite length

$\mathcal{A}$ has **finite length** if for all $E \in \mathcal{A}$, there is a J-H filtration.

Proposition

$\operatorname{rep}_{\mathbb{C}}(Q, R)$ has finite length

Proof

W a f.d. representation of Q. W is simple or there is a SES

$$0 \longrightarrow S \longrightarrow W \longrightarrow Q \longrightarrow 0$$

Repeat and we get B simple and $f : W \longrightarrow B$ surjective.

$$W^1 = \ker(W \longrightarrow B) \subset W \qquad \square$$

Grothendieck group of $\mathrm{rep}_{\mathbb{C}}(Q, R)$

The **Grothendieck group** $K(\mathcal{A})$ is the free abelian group generated by isoclasses mod $C = A + B$ if exists a SES in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

Example: $K(\mathrm{Vec}_{\mathbb{C}}) = \mathbb{Z}$.

Theorem

Let (Q, R) an acyclic bound quiver. Then the Grothendieck group is generated by the vertex set of Q, i.e.,

$$K(\mathcal{A}) = \mathbb{Z}^{Q_0}$$

The comparison

$\mathrm{rep}_{\mathbb{C}}(Q, R)$ is way simpler than $\mathrm{Coh}(X)$

- Simple objects in $\mathrm{Coh}(X)$ are skyscraper sheaves of $x \in X$,
- It is impossible to obtain a filtration by finitely many of these.
- Only sheaves of dimension 0 (supported in dimension 0).
- Thus $\mathrm{Coh}(X)$ cannot have finite length.

The Baer-Bondal theorem

Motivation

Describe $D^b(\mathrm{Coh}(X))$ *using* $D^b(\mathrm{rep}_{\mathbb{C}}(Q, R))$.

Maybe we can make computations on $D^b(\mathrm{Coh}(X))$ *using* Q*?*

Generators of triangulated categories

Let $\mathcal{D}$ be a triangulated category.

- $\mathcal{D}$ is **thick** if its closed under
  - isomorphisms,
  - shifts,
  - cones,
  - direct summands.
- The **eppaise envelope** of E on $\mathcal{D}$ os the smallest thick $\triangle$ subcategory of $\mathcal{D} \supset E$.
- If $\langle E\rangle_{\mathcal{D}} = \mathcal{D}$, then E **generates clasically** $\mathcal{D}$.

Tilting sheaves

We remember that

$$\operatorname{gdim}(A) = \max\left(\operatorname{pdim}(M) \mid M \in \operatorname{Mod}(A)\right)$$

Definition

$T \in \operatorname{Coh}(X)$ is a **tilting sheaf** if

T1. $A = \operatorname{End}_{\mathcal{O}_X}(T)$ has gdim $< +\infty$,

T2. $\operatorname{Ext}^k_{\mathcal{O}_X}(T, T) = 0$ for all $k > 0$,

T3. T clasically generates $D^b(\operatorname{Coh}(X))$

Baer-Bondal, at least

Theorem (Baer, 1988; Bondal, 1990)

Let T be a tilting sheaf on a smooth projective variety X, with $A = \mathrm{End}_{\mathcal{O}_X}(T)$ its associated tilting algebra, then

$$F(-) := \hom_{\mathcal{O}_X}(T, -) : \mathrm{Coh}(X) \longrightarrow \mathrm{mod}(A^{\mathrm{op}})$$
$$G(-) := - \otimes_A T : \mathrm{mod}(A^{\mathrm{op}}) \longrightarrow \mathrm{Coh}(X)$$

induce **equivalences of** $\triangle$d categories:

$$RF(-) : R\hom_{\mathcal{O}_X}(T, -) : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{mod}(A^{\mathrm{op}}))$$
$$LG(-) := - \otimes_A^L T : D^b(\mathrm{mod}(A^{\mathrm{op}})) \longrightarrow D^b(\mathrm{Coh}(X))$$

Sketch of the proof

1. Existence of the right derived functor RF,
2. RF is defined in the categories we want,
3. Existence of the left derived functor LG,
4. LG is defined in the categories we want,
5. RF and LG are in fact inverses.

Existence of RF

$\hom_{\mathcal{O}_X}$ becomes a right $A-$module by composition.

As $\hom(T,-)$ is covariant left-exact and $\mathrm{QCoh}(X)$ has enough injetives, there is a right derived functor

$$RF(-) = R\hom_{\mathcal{O}_X}(T,-) : D^b(\mathrm{QCoh}(X)) \longrightarrow D(\mathrm{Mod}(A^{\mathrm{op}}))$$

Defined in the right sense

What are the cohomologies of the image?

$$H^i(R\,\mathrm{hom}(T,E)) = R^i\,\mathrm{hom}(T,E) = \mathrm{Ext}^i(T,E)$$

As X is smooth, then $\mathrm{Ext}^i = 0$ for $i > \dim X, i < 0$. Thus RF lies in $D^b(\mathrm{Mod}(A^{\mathrm{op}}))$. $D^b(\mathrm{Coh}(X)) \cong \mathcal{D} \subset D^b(\mathrm{QCoh}(X))$ with coherent cohomologies, we can restrict

$$RF\Big|_{D^b(\mathrm{Coh}(X))}$$

If $E \in \mathrm{Coh}(X)$, $\mathrm{Ext}^i(T,E)$ is f.d. as a $\mathbb{C}-$vector space, hence as an $A-$module. Thus,

$$\text{cohomologies lie in } \mathrm{mod}(A^{\mathrm{op}})$$

Existence of LG

$$RF : D^b(\mathsf{Coh}(X)) \longrightarrow D^b(\mathsf{mod}(A^{\mathsf{op}}))$$

- $\mathsf{Mod}(A^{\mathsf{op}})$ has enough projectives,
- $- \otimes_A T$ is right exact,

then we get a left-derived functor

$$LG(-) = - \otimes_A^L T : D^b(\mathsf{Mod}(A^{\mathsf{op}})) \to D(\mathsf{QCoh}(X))$$

Defined in the right sense

$B \in \mathrm{Mod}(A)$, then its cohomology sheaves:

$$H^i(B \otimes^L T) = \mathrm{Tor}_A^{-i}(B, T)$$

As A has finite gdim, then this is 0 after a finite number.

If we restrict to f.g. A—modules, the cohomologies are coherent, giving

$$LG(-) = - \otimes_A^L T : D^b(\mathrm{mod}(A^{\mathrm{op}})) \longrightarrow D^b(\mathrm{Coh}(X))$$

In fact inverses

As T satisfies T2,

$$\begin{aligned} RF \circ LG(A) &= RF(A \otimes_A^L T) \\ &= R\hom_{\mathcal{O}_X}(T, A \otimes_A^L T) \\ &= \hom_{\mathcal{O}_X}(T, T) = A \end{aligned}$$

We can show that A clasically generates $D^b(\mathrm{mod}(A^{\mathrm{op}}))$, thus the composition is the identity.

In fact inverses

T satisfies T3, we can do a similar argument and as T clasically generates $D^b(\text{Coh}(X))$, the composition is the identity.

Then RF and LG are inverses.

Applications and Exceptional collections

The natural map

Let $E \in D^b(\mathcal{A})$, then

$$[E] := \sum_i (-1)^i [E^i] \in K(\mathcal{A})$$

gives a well defined map

$$[-] : D^b(\mathcal{A}) \longrightarrow K(\mathcal{A})$$

such that $[E + F] = [E] + [F]$.

Bilinear forms

1. $K(\mathrm{Coh}(X))$ admits one:

$$\langle [E], [F] \rangle := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathcal{O}_X}(E, F)$$

2. For $K(\mathrm{mod}(A))$, if $\mathrm{gdim}\, A < +\infty$,

$$\langle [M], [N] \rangle := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(M, N)$$

Isomorphism of Grothendieck groups

Proposition

T a tilting sheaf on X, $A = \mathrm{End}_{\mathcal{O}_X}(T)$, then

$$[RF(-)] : K(\mathrm{Coh}(X)) \longrightarrow K(\mathrm{mod}(A^{\mathrm{op}}))$$
$$[RF(-)] = \sum_i (-1)^i \,\mathrm{Ext}^i_{\mathcal{O}_X}(T, -)$$

is an isomorphism that preserves the natural bilinear forms on each side.

Necessary condition

In particular,

Corollary

If X is a smooth projective variety that admits a tilting sheaf, then $K(\mathrm{Coh}(X))$ is finitely generated and free.

Now our question becomes, which varieties admit tilting bundles?

(Full and) Exceptional collections

Let $\mathcal{D}$ be a triangulated category.

1. $E \in \mathcal{D}$ is **exceptional** if

$$\hom_{\mathcal{D}}(E, E) = \mathbb{C}, \quad \hom_{\mathcal{D}}(E, E[k]) = 0 \; \forall k \neq 0.$$

2. A sequence $(E_0, \ldots, E_m)$ of exceptional objects is **exceptional** if

$$R\hom_{\mathcal{D}}(E_i, E_j) = 0 \text{ for all } i > j,$$

3. it is **strongly exceptional** if we also have $\hom_{\mathcal{D}}(E_i, E_j[k]) = 0$ for $i < j, k \neq 0$,
4. and **full** if $E_0, \ldots, E_m$ generate $\mathcal{D}$ as a $\triangle$ category.

Tilting bundles and exceptional collections

Proposition

Let T be a locally-free sheaf on X such that $T = \bigoplus_{i=0}^{m} E_i$ is a decomposition into line bundles.

1. If T satisfies T1 and T2, then

$$(E_0, \dots, E_m) \text{ is a } \textbf{strong exceptional} \text{ collection}$$

2. If T satisfies T1, T2 and T3, then

$$(E_0, \dots, E_m) \text{ is a } \textbf{full strong exceptional} \text{ collection}$$

Every full strong exceptional collection defines a tilting sheaf.

Bibliography I

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Thanks!