Phantoms on Rational Surfaces

Amal Mattoo

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Overview

Definition

A *(geometric) phantom* is an admissible subcategory $\mathcal{A} \subset D^b(X)$ for X smooth projective such that $K_0(\mathcal{A}) = 0$ and $\mathrm{HH}_\bullet(\mathcal{A}) = 0$.

- K_0, $\mathrm{HH}_\bullet$, $\mathrm{HH}^\bullet$, (disproven) conjectures, and phantoms.
- Kuznetsov's theory of heights, Krah's construction, extension to other rational surfaces.
- Studying phantoms using $\mathrm{HH}^\bullet$ and spectral sequence for $\mathrm{Hom}^\bullet(i^*-, i^*-)$.

Hochschild (Co)homology

For a dg-cat $\mathcal{C}$: $\mathsf{HH}_\bullet = \mathsf{Tor}^\bullet_{\mathcal{C}^{\mathsf{op}} \otimes \mathcal{C}}(\mathcal{C}, \mathcal{C})$, $\mathsf{HH}^\bullet = \mathsf{Ext}^\bullet_{\mathcal{C}^{\mathsf{op}} \otimes \mathcal{C}}(\mathcal{C}, \mathcal{C})$.
For a triangulated cat., take a dg-enhancment.

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For $D^b(X)$:

$$\mathrm{HH}_\bullet = H^\bullet(X \times X, \Delta_*\mathcal{O}_X \otimes \Delta_*\mathcal{O}_X) = \bigoplus_{p=1}^{n} H^{\bullet+p}(X, \Omega_X^p)$$

$$\mathrm{HH}^\bullet = \mathrm{Hom}^\bullet_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) = \bigoplus_{p=0}^{n} H^{\bullet-p}(X, \bigwedge^p T_X)$$

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If $D^b(X) = \langle \mathcal{A},\mathcal{B}\rangle$, then for $\mathcal{P}_\mathcal{B} \to \Delta_*\mathcal{O}_X \to \mathcal{P}_\mathcal{A}$ in $D^b(X\times X)$,

$$\mathrm{HH}_\bullet(\mathcal{A}) = \mathrm{H}^\bullet(X\times X, \mathcal{P}_\mathcal{A}\otimes\mathcal{P}_\mathcal{A}^T), \quad \mathrm{HH}^\bullet(\mathcal{A}) = \mathrm{Hom}^\bullet_X(\mathcal{P},\mathcal{P})$$

Note: $\mathcal{P}_\mathcal{A}$ is FM kernel for $i^*: D^b(X)\to\mathcal{A}$.

Semi-orthogonal decompositions and (disproven) conjectures

K_0 and $\mathsf{HH}_\bullet$ are additive:
$\mathcal{C} = \langle \mathcal{A}_i \rangle_i \implies K_0(\mathcal{C}) = \bigoplus_i K_0(\mathcal{A}_i), \quad \mathsf{HH}_\bullet(\mathcal{C}) = \bigoplus_i \mathsf{HH}_\bullet(\mathcal{A}_i).$
$\mathsf{HH}^\bullet$ is NOT, but
$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle \implies \mathsf{HH}^\bullet(\mathcal{C}) \to \mathsf{HH}^\bullet(\mathcal{A}) \oplus \mathsf{HH}^\bullet(\mathcal{B}) \to \mathsf{Ext}^\bullet(\phi, \phi)$
where $\phi : \mathcal{B} \to \mathcal{A}$ is the gluing functor $i^!|_{\mathcal{B}}$.

Semi-orthogonal decompositions and (disproven) conjectures

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Conjecture (Kuznetsov 2009)

If $\mathcal{A} \subset D^b(X)$ admissible with $\mathrm{HH}_\bullet(\mathcal{A}) = 0$, then $\mathcal{A} = 0$.

Conjecture (Kuznetsov 2014)

If $\mathcal{C}$ has a full exceptional collection $\langle \mathcal{E}_1, ..., \mathcal{E}_n \rangle$, then every exceptional collection of length n is full.

Meta conjecture: every conjecture about derived categories is false!

Phantom categories

Definition

A *(geometric) phantom* is an admissible subcategory $\mathcal{A} \subset D^b(X)$ for X smooth projective such that $K_0(\mathcal{A}) = 0$ and $\mathrm{HH}_\bullet(\mathcal{A}) = 0$.

Example (GO 2013, BGvBKS 2015)

On general type surfaces and their products.

Example (Krah 2023)

On $\mathrm{Bl}_{10\mathrm{pts}}\mathbb{P}^2$.

Example (KKLLMMPRV 2025, MXY 2025)

On $\mathrm{Bl}_{11\mathrm{pts}}\mathbb{P}^2$, $\mathrm{Bl}_{9\mathrm{pts}}\mathbb{F}_2$, $\mathrm{Bl}_{10\mathrm{pts}}\mathbb{P}^2$ (new?).

Idea: find a maximal exceptional collection that is not full.

Kuznetsov's normal Hochschild cohomology and heights

How to tell $\mathcal{B} = \langle \mathcal{E}_1, ..., \mathcal{E}_n \rangle \subset D^b(X)$ isn't full? Letting $\mathcal{A} = \mathcal{B}^\perp$,

$$\mathrm{NHH}^\bullet(\mathcal{B}, X) \to \mathrm{HH}^\bullet(X) \to \mathrm{HH}^\bullet(\mathcal{A})$$

$\mathrm{NHH}^\bullet$ defined via dg-categories, computed by a spectral sequence.

Definition

The *height* of $\mathcal{B}$ is the minimal h such that $\mathrm{NHH}^h(\mathcal{B}, X) \neq 0$.
The *pseudoheight* is minimal h such that $E_1^{p,q} \neq 0$ with $p + q = h$.

Input is non-trivial degrees of $\mathrm{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j)$ and $\mathrm{Hom}^\bullet(\mathcal{E}_i, S^{-1}\mathcal{E}_j)$.

Theorem (Kuznetsov 2014)

If ph > 0, *then* $\mathcal{A} \neq 0$.

Proof: h $\geq$ ph, so h $> 0 \implies \mathrm{HH}^0(X) \hookrightarrow \mathrm{HH}^0(\mathcal{A}) \implies \mathcal{A} \neq 0$.

Krah's construction

Let $X = \mathrm{Bl}_{10\mathrm{pts}}\mathbb{P}^2$, points in general position. Note $K_X^2 = -1$.
Starting with $\langle \mathcal{O}_X, \mathcal{O}_X(E_1) ..., \mathcal{O}_X(E_{10}), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$,
apply reflection $\iota : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$, $K_X \mapsto K_X$, $K_X^\perp \mapsto -K_X^\perp$:

$$D_i := \iota(E_i) = -6H + 2\sum_{j=1}^{10} E_j - E_i, \quad F := \iota(H) = -19H + 6\sum_{j=1}^{10} E_i$$

Theorem (Krah 2023)

$\mathcal{B} := \langle \mathcal{O}_X, \mathcal{O}_X(D_1) ..., \mathcal{O}_X(D_{10}), \mathcal{O}_X(F), \mathcal{O}_X(2F) \rangle$ is an exceptional collection and $\mathcal{A} := \mathcal{B}^\perp$ is a phantom.

Proof: $\mathrm{Hom}^\bullet$ vanishing by case of SHGH conjecture/computer.
And $\dim \mathrm{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = \chi(\mathcal{E}_i, \mathcal{E}_j)[2]$ for $i < j$, so $h > 0$.
Also found $\mathrm{HH}^2(\mathcal{A}) \cong H^1(\mathcal{T}_X) \cong \mathbb{C}^{12}$.

Other rational surfaces

Theorem (KKLLMMPRV 2025)

Exist reflections on Pic *of* $\mathrm{Bl}_{11\mathrm{pts}}\mathbb{P}^2$ *and* $\mathrm{Bl}_{9\mathrm{pts}}\mathbb{F}_2$ *sending standard full exceptional collections to non-full exceptional collections. The resulting phantoms are distinct from each other and Krah's.*

$$\mathrm{HH}^2(\mathcal{A}_{\mathrm{Bl}_{11\mathrm{pts}}\mathbb{P}^2}) \hookleftarrow \mathrm{HH}^2(\mathrm{Bl}_{11\mathrm{pts}}\mathbb{P}^2) \cong H^1(\mathcal{T}_{\mathrm{Bl}_{11\mathrm{pts}}\mathbb{P}^2}) \cong \mathbb{C}^{14}.$$
$$\mathrm{HH}^2(\mathcal{A}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{F}_2}) = \mathrm{HH}^2(\mathrm{Bl}_{9\mathrm{pts}}\mathbb{F}_2) \cong H^1(\mathcal{T}_{\mathrm{Bl}_{9\mathrm{pts}}\mathbb{F}_2}) \cong \mathbb{C}^{13}.$$

Theorem (KKLLMMPRV 2025)

There is another reflection on $\mathrm{Pic}(\mathrm{Bl}_{10\mathrm{pts}}(\mathbb{P}^2))$ *yielding a phantom.*

Reflection can be more general across a plane containing K_X.
Pseudoheight insufficient to distinguish this phantom from Krah's.

Conjecture (KKLLMMPRV 2025)

Exists a phantom on $\mathrm{Bl}_{d\mathrm{pts}}\mathbb{F}_n$ *for* $d \geq 6 + \max\{3, n\}$.

Spectral sequence for $\mathrm{Hom}(i^*-, i^*-)$

To study objects of $\mathcal{A}$, compute left adjoint i^* to $i : \mathcal{A} \hookrightarrow D^b(X)$. For $\mathcal{E}$ exceptional, the left projection for $\langle \mathcal{E}^\perp, \mathcal{E} \rangle$ is given by cone

$$\mathrm{Hom}^\bullet(\mathcal{E}, K) \otimes \mathcal{E} \to K \to i^* K$$

Iterating this, we obtain (compare $\mathrm{NHH}^\bullet$ spectral sequence):

Proposition (M. 2025)

$E_1^{p,q} \implies \mathrm{Hom}^{p+q}(i^*K', i^*K)$ with $E_1^{-p-1,q} =$

$$\bigoplus_{\substack{0 \le a_0 < \ldots < a_p \le n, \\ k_0 + \ldots + k_p + k = q}} \mathrm{Hom}^{k}(K', \mathcal{E}_{a_0}) \otimes \mathrm{Hom}^{k_0}(\mathcal{E}_{a_0}, \mathcal{E}_{a_1}) \otimes \ldots \otimes \mathrm{Hom}^{k_{p-1}}(\mathcal{E}_{a_{p-1}}, \mathcal{E}_{a_p}) \otimes \mathrm{Hom}^{k_p}(\mathcal{E}_{a_p}, K)$$

for $p \ge 0$ and $E_1^{0,q} = \mathrm{Hom}^q(K', K)$, with d_1 signed composition.

Projections of skyscraper sheaves to Krah's phantom

$$E_1^{-p-1,q} =$$

$$\bigoplus_{\substack{0\leq a_0<\ldots<a_p\leq n,\\ k_0+\ldots+k_p+k=q}} \mathbf{Hom}^k(\kappa(x),\mathcal{E}_{a_0})\otimes\mathbf{Hom}^{k_0}(\mathcal{E}_{a_0},\mathcal{E}_{a_1})\otimes\ldots\otimes\mathbf{Hom}^{k_{p-1}}(\mathcal{E}_{a_{p-1}},\mathcal{E}_{a_p})\otimes\mathbf{Hom}^{k_p}(\mathcal{E}_{a_p},\kappa(x))$$

for $p \geq 0$ and $E_1^{0,q} = \mathbf{Hom}^q(\kappa(x),\kappa(x))$.

$$\ddots \bigoplus_{i<j} \mathbf{Hom}^2(\kappa(x),\mathcal{E}_i)\otimes\mathbf{Hom}^2(\mathcal{E}_i,\mathcal{E}_j)\otimes\mathbf{Hom}^0(\mathcal{E}_j,\kappa(x))$$

$$\bigoplus_{i} \mathbf{Hom}^2(\kappa(x),\mathcal{E}_i)\otimes\mathbf{Hom}^0(\mathcal{E}_i,\kappa(x)) \to \mathbf{Hom}^2(\kappa(x),\kappa(x))$$

$$\mathbf{Hom}^1(\kappa(x),\kappa(x))$$

$$\mathbf{Hom}^0(\kappa(x),\kappa(x))$$

Proposition

For any $x \in X$,

$$\mathrm{Hom}^\bullet(i^*\kappa(x), i^*\kappa(x)) = \mathbb{C}^1[0]\oplus\mathbb{C}^{14}[-1]\oplus\mathbb{C}^{92}[-2]\oplus\mathbb{C}^{139}[-3]\oplus\mathbb{C}^{60}[-4]$$

Compositions, deformations, and $X \to \mathcal{A}$

Since negative Homs vanish and $\mathrm{Hom}^0 \cong \mathbb{C}$, $i^*\kappa(x) \in s\mathcal{M}(\mathcal{A})$ the $\mathbb{G}_m$-gerbe of *simple universally gluable objects*.
Deformations of $i^*\kappa(x)$ from Hom^1, obstructions from $\circ$ via

$$\mathrm{Hom}^0(\mathcal{E}_j, \kappa(x)) \otimes \mathrm{Hom}^2(\kappa(x), \mathcal{E}_i) \to \mathrm{Hom}^2(\mathcal{E}_j, \mathcal{E}_i)$$

For generic $x \in X$, turns out all are obstructed but those from $\mathcal{T}_x X$!

Proposition

An irreducible component of $sM(\mathcal{A})$ is birational to X.

But there is a special locus where $i^*\kappa(x)$ could deform more.

Theorem

The only irreducible effective divisors supported in the special locus of X are $|-K_X + E_i|$ for $1 \le i \le 10$. Thus, $i\colon \mathcal{A} \hookrightarrow D^b_{\mathrm{Coh}}(X)$ characterizes the blowdown map $\pi\colon X \to \mathbb{P}^2$.

Other objects in $\mathcal{A}$

Strong generator $\mathcal{Q} \in D^b(X)$ with $\mathrm{Hom}^\bullet(i^*\mathcal{Q}, i^*\mathcal{Q}) = 0$ for $\bullet < 0$.

Theorem

Exists a co-connective dg-algebra A with $D^b(A)$ a phantom.

$\iota : C \to X$, $C \in |-nF|$ for $n \geq 3$, $\mathcal{L} \in \mathrm{Pic}^{g-1}(C)$, $\mathcal{P} := i^*\iota_*\mathcal{L}$.

Proposition

$\mathcal{H}^i(\mathcal{P})$ is zero for $i \neq 0, 1$, and $\mathcal{P} \in s\mathcal{M}(\mathcal{A})$ with

$$0 \to H^1(C, \mathcal{O}_C) \to \mathrm{Hom}^1(\mathcal{P}, \mathcal{P}) \to H^0(C, \mathcal{N}_{X/C}) \to 0$$

Deformations of $\mathcal{P}$ recover C and maybe X!
But how to intrinsically identify $\mathcal{P} \in \mathcal{A}$?

Alternate approach to Hochschild cohomology

If $\mathcal{A} = \langle \mathcal{E}_i \rangle_i^{\perp}$ and $D^b(X) = \langle \mathcal{L}_j \rangle_j$, then $\mathcal{A} \boxtimes D^b(X) = \langle \mathcal{E}_j \boxtimes \mathcal{L}_i \rangle^{\perp}$.
Spectral sequence with $\imath \colon \mathcal{A} \boxtimes D^b(X) \hookrightarrow D^b(X \times X)$ computes

$$\mathrm{Hom}^\bullet(\imath^* \Delta_* \mathcal{O}_X, \imath^* \Delta_* \mathcal{O}_X) = \mathrm{HH}^\bullet(\mathcal{A})$$

Complicated in practice, but can access product structure on $\mathrm{HH}^\bullet$.

Proposition

For $\mathcal{A}$ Krah's phantom, the product on $\mathrm{HH}^\bullet(\mathcal{A})$ is trivial.

Thank you for listening!

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