19 May 2026

Noncommutative Resolutions of Kleinian Singularities

Derived
Obsessed
Graduate
Students

1. Recap: the McKay correspondence
2. More resolutions ⇜ jt. with Ruth Wye
3. Monoidal structures (sketchy) ⇜ ongoing with Austin Hubbard

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__1.__ $\Gamma < SL_2(\mathbb{C})$ finite subgroup

$$\rightsquigarrow \Gamma \circlearrowright \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$$

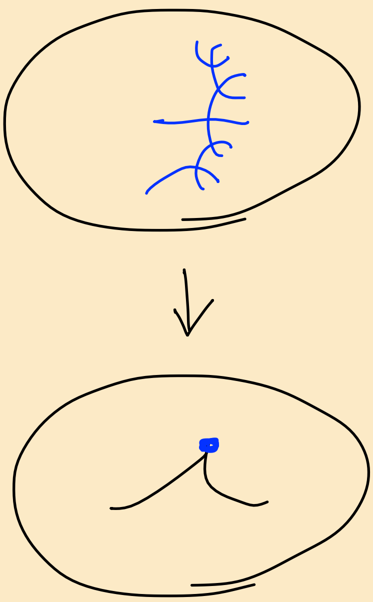
symplectic

$$\boxed{\mathbb{A}^2/\Gamma = \mathrm{Spec}(\mathbb{C}[x,y]^\Gamma)}$$

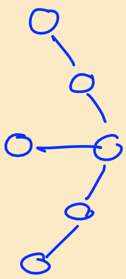
↑ Kleinian singularity

Y

minimal resolution



$\mathbb{A}^2/\Gamma$



{ADE Dynkin diagrams} ⟷ {finite $\Gamma < SL(2,\mathbb{C})$ up to conj.}

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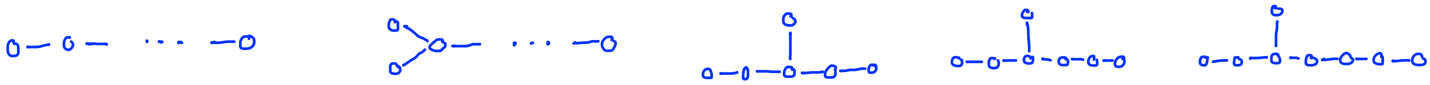
A_n $(n \geq 1)$ D_n $(n \geq 4)$ E_6 E_7 E_8



$$\mathrm{Coh}([\mathbb{A}^2/\Gamma]) = \mathrm{Coh}_\Gamma(\mathbb{A}^2)$$

$$\begin{tikzcd}
Y \arrow[rr, bend left, dashed, leftrightarrow, "\text{McKay correspondence}"] \arrow[dr] & & {[\mathbb{A}^2/\Gamma]} \arrow[dl] \\
& \mathbb{A}^2/\Gamma &
\end{tikzcd}$$

both "crepant"

<u>Theorem</u> [Kapranov – Vasserot ~2000]

$$\boxed{D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma]))}$$

<u>Conjecture</u> [Bondal, Kawamata] The same is true for any two crepant resolutions of a given Gorenstein singularity.

Noncommutative algebras

$\exists$ a n.c. algebra Π with $Z(\Pi) = \mathbb{C}[x,y]^\Gamma$ s.t.

$$\boxed{\mathrm{mod}(\Pi) \simeq \mathrm{Coh}([\mathbb{A}^2/\Gamma]).}$$

Π is a n.c. resolution [van den Bergh] of $\mathbb{A}^2/\Gamma$.

$$\boxed{D^b(\mathrm{Coh}(Y)) \xrightarrow[\mathrm{RHom}(V^*,-)]{\sim} D^b(\mathrm{mod}(\Pi))}$$

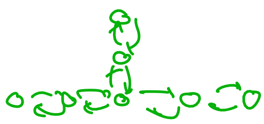
tilting equivalence

$$D^b(\mathrm{mod}(\Pi)) \cong D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma]))$$

where V is a v.b. on Y with $\mathrm{End}(V) \cong \Pi$.

Π is usually:

preprojective algebra



skew-group algebra

$$\mathbb{C}[x,y] \otimes_{\mathbb{C}} \mathbb{C}\Gamma$$

$$\gamma \cdot f = \gamma(f) \cdot \gamma$$

$\gamma \in \Gamma$, $f \in \mathbb{C}[x,y]$

2.

Example: (Types A_n (n odd) or any D_n, E_n)

$$\Gamma \circlearrowright \mathbb{A}^2 \Rightarrow \Gamma \circlearrowright \mathbb{P}^1 \Rightarrow \Gamma \circlearrowright \underbrace{T^*\mathbb{P}^1}_{\text{symplectic}}$$

Let $\mathbb{P}\Gamma$ be the image of Γ in $PGL_2(\mathbb{C})$. Then $\mathbb{P}\Gamma = \Gamma/\{\pm 1\}$. We have a crepant resolution

$$\boxed{T^*\mathbb{P}^1 \longrightarrow \mathbb{A}^2/\{\pm 1\}},$$

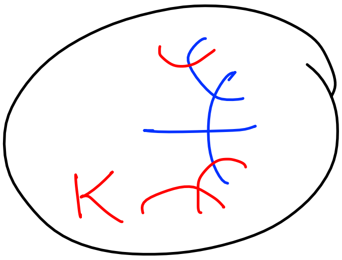
equivariant with respect to $\mathbb{P}\Gamma$. Hence, we have a crepant resolution

$$\boxed{[T^*\mathbb{P}^1/\mathbb{P}\Gamma] \longrightarrow \mathbb{A}^2/\Gamma}.$$

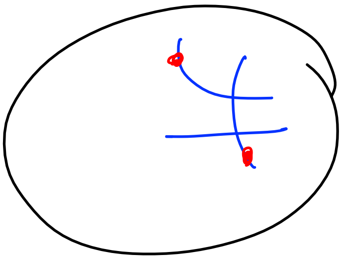
Theorem [Brav] ("projective McKay correspondence")

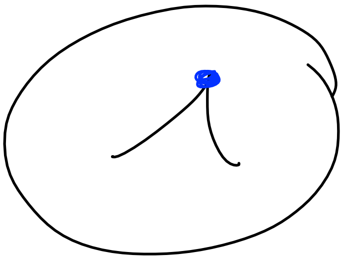
$$\boxed{D^b([T^*\mathbb{P}^1/\mathbb{P}\Gamma]) \simeq D^b([\mathbb{A}^2/\Gamma])}$$

More generally . . .



crepant partial resolution





$$\begin{tikzcd}
Y \arrow[dr, "h"'] & & \mathcal{Y}_K \arrow[dl] & {[\mathbb{A}^2/\Gamma]} \arrow[ddl] \\
& Y_K \arrow[dr, "g"'] & & \\
& & \mathbb{A}^2/\Gamma &
\end{tikzcd}$$

<u>Theorem</u> [Chen-Tseng]

$$\boxed{D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Coh}(\mathcal{Y}_K)) \simeq D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma]))}$$

<u>Theorem</u> [B-Wye] Set $\mathcal{E}_K := h_* \mathcal{E}nd(V)$, (on Y)

then $\boxed{\mathrm{mod}(\mathcal{E}_K) \simeq \mathrm{Coh}(\mathcal{Y}_K)}$ (over Y_K).

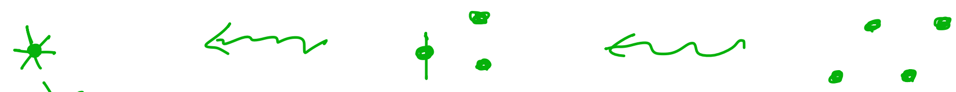
$$\begin{tikzcd}
D^b(\mathrm{Coh}(Y)) \arrow[r, "\sim"] \arrow[d, "V\otimes -"', "\wr\ \text{Morita}"] & D^b(\mathrm{Coh}(\mathcal{Y}_K)) \arrow[r, "\sim"] \arrow[d, "\wr\ \text{Morita}"] & D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma])) \arrow[d, "\wr\ \text{Morita}"] \\
D^b(\mathrm{mod}(\mathrm{End}(V))) \arrow[r, "\sim", "Rh_*"'] & D^b(\mathrm{mod}(\mathcal{E}_K)) \arrow[r, "\sim", "Rg_*"'] & D^b(\mathrm{mod}(\Pi))
\end{tikzcd}$$

Y $\qquad$ Y_K $\qquad$ $\mathbb{A}^2/\Gamma$ [van den Bergh]

3.

<u>Theorem</u> [Fukuyama-Iwanari]
A "nice" DM-stack $\mathcal{Y}$ can be recovered from $(D^b(\mathcal{Y}), \otimes)$ as $\{ D^b(\mathcal{Y}) \xrightarrow{\text{mon.}} D^b(pt) \}$.

	$K_0(Y) \otimes \mathbb{C}$	$K_0(\mathcal{Y}_k) \otimes \mathbb{C}$	$K_0([\mathbb{A}^2/\Gamma]) \otimes \mathbb{C}$
Spec			



$(V_i - \delta_i)(V_j - \delta_j) = 0$

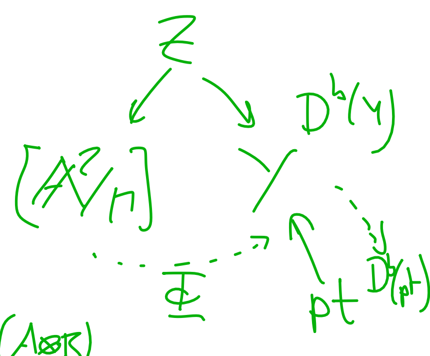
<u>Q:</u> What would be a natural parameter space for this deformation?
Relation with quantum cohomology / K-theory?

<u>Q:</u> How does one "deform" monoidal structures on a derived category?

<u>Problem</u> [Abramovich, Abdelgadir – Segal]:
Relate the different $\mathcal{Y}_k$ via "VGIT."

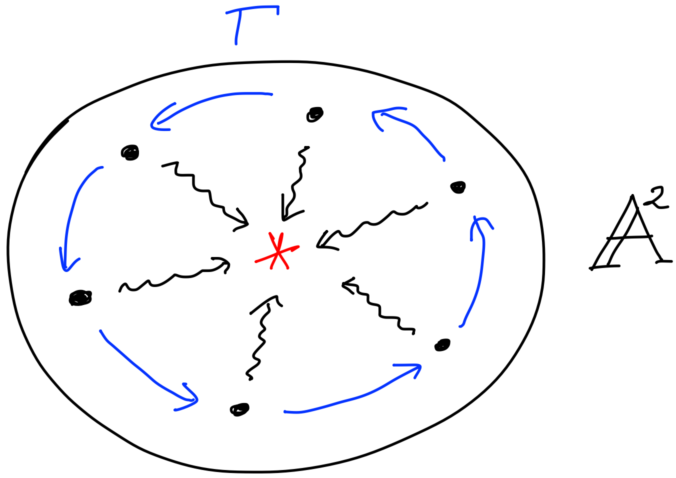
<u>Partial Solution</u> [van de Kreeke]:
parametrize lax monoidal functors

$$D^b([\mathbb{A}^2/\Gamma]) \longrightarrow D^b(pt)$$

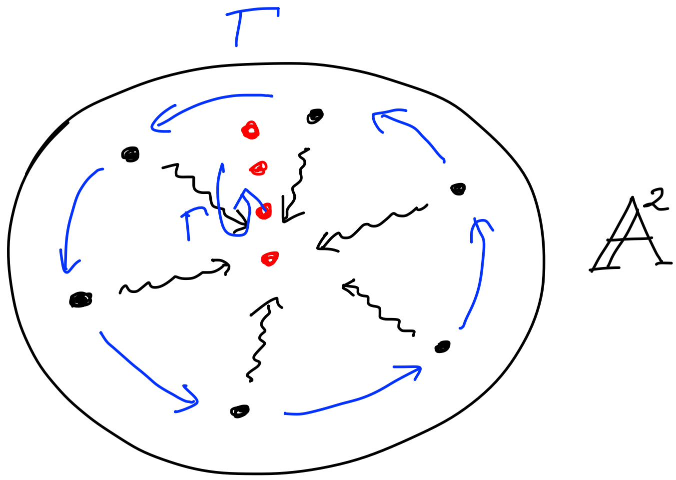


$\Phi(A) \otimes \Phi(B) \to \Phi(A \otimes B)$

$$Y = \mathrm{Hilb}^1([\mathbb{A}^2/\Gamma]) = \Gamma\text{-}\mathrm{Hilb}(\mathbb{A}^2)$$



$$[\mathbb{A}^2/\Gamma]$$



$$\mathcal{Y}_K$$