

19 May  
2026

# Noncommutative Resolutions of Kleinian Singularities

Derived  
Obsessed  
Graduate  
Students

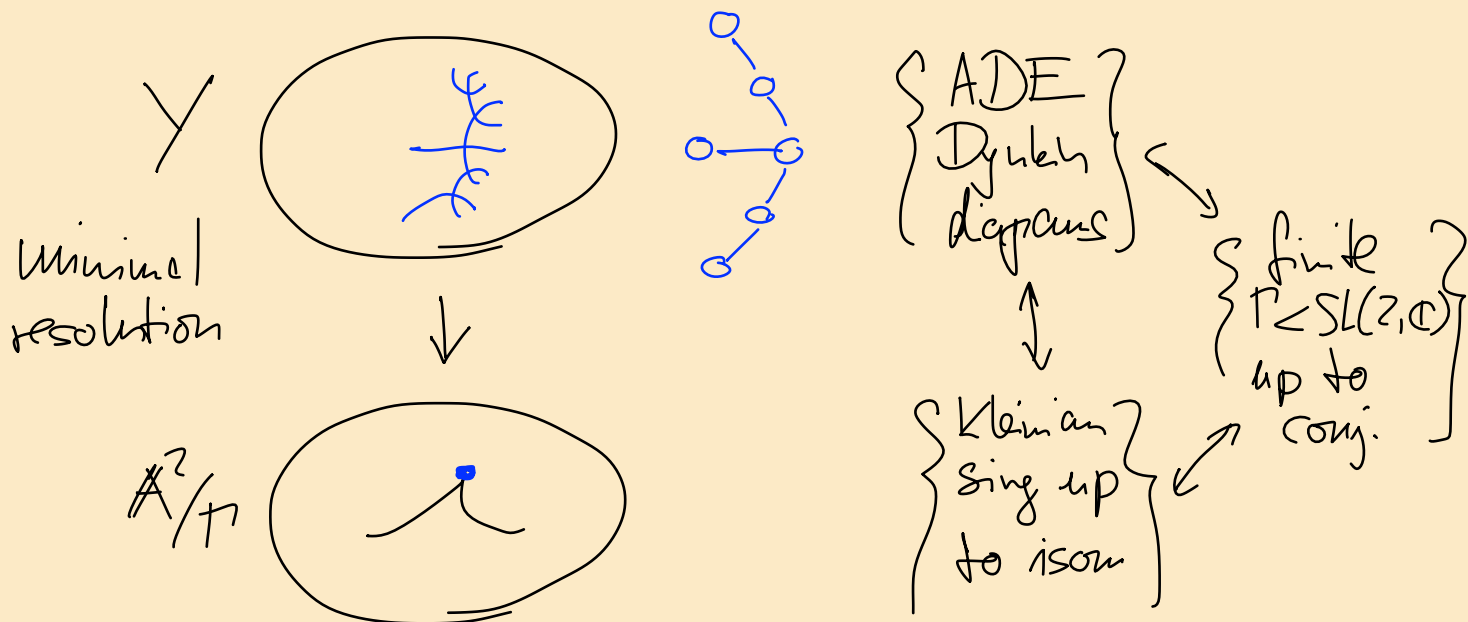
1. Recap: the McKay correspondence
  2. More resolutions ← jt. with Ruth Wye
  3. Monoidal structures (sketchy) ← ongoing with Austin Hubbard
- 

1.  $\Gamma < SL_2(\mathbb{C})$  finite subgroup

$\rightsquigarrow \Gamma \subset \mathbb{A}^2 = \text{Spec}(\mathbb{C}[x,y])$   
symplectic

$\mathbb{A}^2/\Gamma = \text{Spec}(\mathbb{C}[x,y]^\Gamma)$

↑ Kleinian singularity



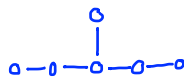
$A_n$  ( $n \geq 1$ )



$D_n$  ( $n \geq 4$ )



$E_6$



$E_7$

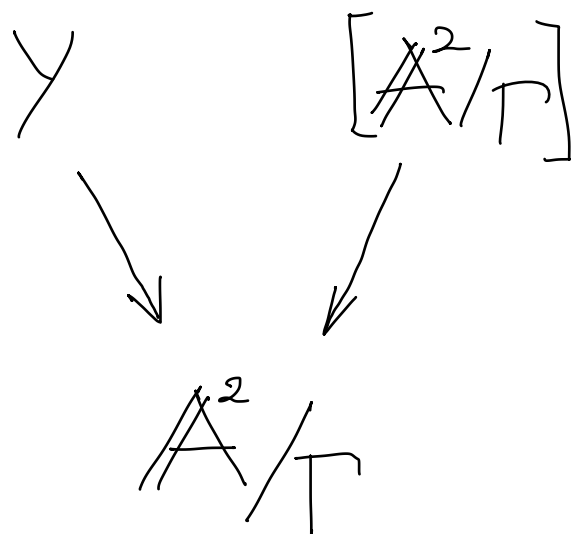


$E_8$



McKay  
Correspondence

$$\text{Coh}([A^2/\Gamma]) = \text{Coh}_\Gamma(A^2)$$



both  
"crepant"

Theorem [Kapranov - Vasserot ~2000]

$$D^b(\text{Coh}(Y)) \simeq D^b(\text{Coh}([A^2/\Gamma]))$$

Conjecture [Bondal, Kawamata] The same is

true for any two crepant resolutions of a given Gorenstein singularity.

# Noncommutative algebras

$\exists$  a n.c. algebra  $\mathbb{T}$  with  $Z(\mathbb{T}) = \mathbb{C}[x, y]^{\Gamma}$  st.

$$\text{mod}(\mathbb{T}) \simeq \text{Coh}(\mathbb{A}^2/\Gamma).$$

$\mathbb{T}$  is a n.c. resolution [vander Bergh] of  $\mathbb{A}^2/\Gamma$ .

$$D^b(\text{Coh}(Y)) \xrightarrow[\text{RH } (V, -)]{\sim} D^b(\text{mod}(\mathbb{T}))$$

tilting equivalence

$$\rightarrow D^b(\text{Coh}(\mathbb{A}^2/\Gamma))$$

where  $V$  is a v.b. on  $Y$  with  $\text{End}(V) \cong \mathbb{T}$ .

$\mathbb{T}$  is usually:

preprojective algebra

skew-group algebra

$$\begin{matrix} & \mathfrak{g} \cdot f & \\ \uparrow & & \uparrow \\ \Gamma & & \mathbb{C}[x, y] \end{matrix}$$

2.

Example: (Types  $A_n$  ( $n$  odd) or any  $D_n, E_n$ )

$$T \hookrightarrow \mathbb{A}^2 \Rightarrow T \hookrightarrow \mathbb{P}^1 \Rightarrow T \hookrightarrow T^*\mathbb{P}^1$$

Let  $\mathbb{P}\Gamma$  be the image of  $\Gamma$  in  $\mathrm{PGL}_2(\mathbb{C})$ . Then

$\mathbb{P}\Gamma = T / \{\pm 1\}$ . We have a crepant resolution

$$\boxed{T^*\mathbb{P}^1 \longrightarrow \mathbb{A}^2 / \{\pm 1\}},$$

equivariant with respect to  $\mathbb{P}\Gamma$ . Hence,

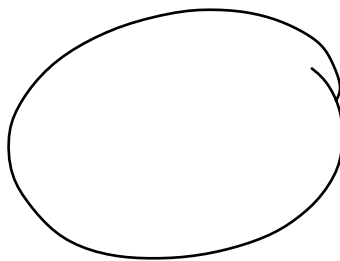
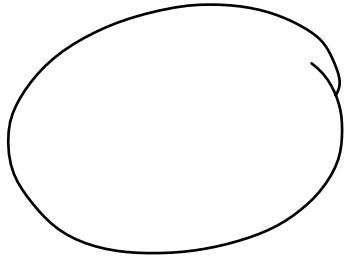
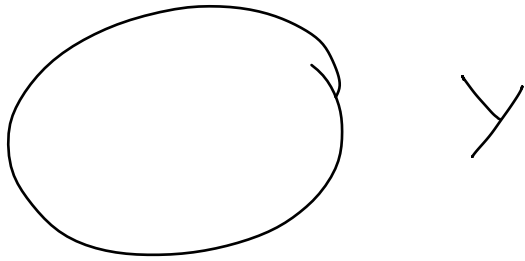
We have a crepant resolution

$$\boxed{[T^*\mathbb{P}^1 / \mathbb{P}\Gamma] \longrightarrow \mathbb{A}^2 / \Gamma}.$$

Theorem [Brau] ("projective McKay correspondence")

$$\boxed{D^b([T^*\mathbb{P}^1 / \mathbb{P}\Gamma]) \simeq D^b([\mathbb{A}^2 / \Gamma])}$$

More generally ...



$\mathbb{A}^2/\Gamma$

Theorem [Chen-Tsung]

$$D^b(\text{Coh}(Y)) \simeq D^b(\text{Coh}(Y_K)) \simeq D^b(\text{Coh}(\mathbb{A}^2/\Gamma))$$

Theorem [B-Wye] Set  $\mathcal{E}_K := h_* \text{End}(V)$ ,

then  $\text{mod}(\mathcal{E}_K) \simeq \text{Coh}(Y_K)$  (over  $Y_K$ ).

$$D^b(\text{Coh}(Y)) \xrightarrow{\sim} D^b(\text{Coh}(Y_K)) \xrightarrow{\sim} D^b(\text{Coh}(\mathbb{A}^2/\Gamma))$$

$\downarrow$                        $\downarrow$                        $\downarrow$

$$D^b(\text{mod}(\text{End}(V))) \xrightarrow{\sim} D^b(\text{mod}(\mathcal{E}_K)) \xrightarrow{\sim} D^b(\text{mod}(\mathbb{T}))$$

3.

Theorem [Fukuyama-Iwanari]

A "nice" DM-stack  $\mathcal{Y}$  can be recovered from  $(D^b(\mathcal{Y}), \otimes)$  as  $\{D^b(\mathcal{Y}) \xrightarrow{\text{mor.}} D^b(\text{pt})\}$ .

	$K_0(\mathcal{Y}) \otimes \mathbb{C}$	$K_0(\mathcal{Y}_k) \otimes \mathbb{C}$	$K_0([\mathbb{A}^2/\Gamma]) \otimes \mathbb{C}$
Spec	$(V_i - \delta_i)(V_j - \delta_j) = 0$		

Q: What would be a natural parameter space for this deformation?

Relation with quantum cohomology / K-theory?

Q: How does one "deform" monoidal structures on a derived category?

Problem [Abramovich, Abdelgadir - Segal]:

Relate the different  $\mathcal{Y}_k$  via "VGIT".

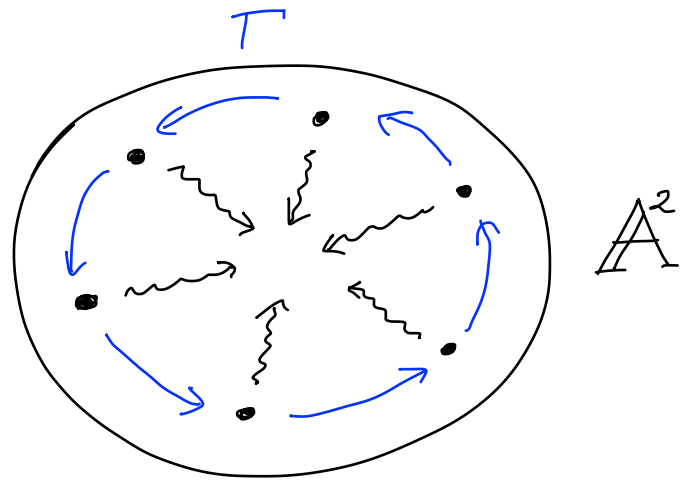
Partial Solution [van de Kreeke]:

parametrize lax monoidal functors

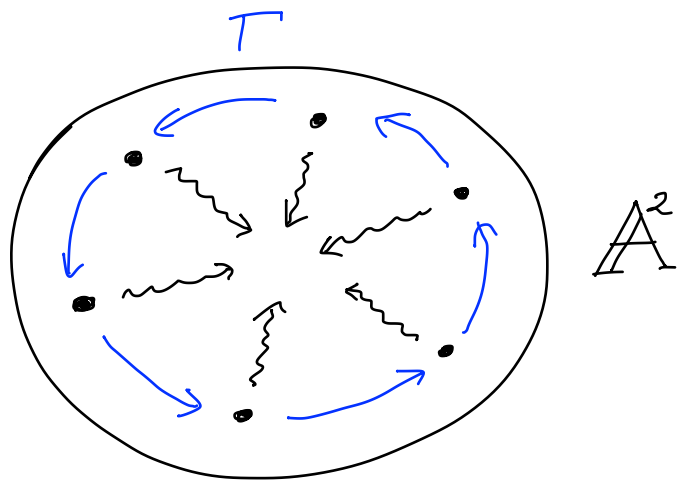
$$D^b([\mathbb{A}^2/\Gamma]) \longrightarrow D^b(\text{pt})$$

$$Y = \text{Hilb}^1([A^2/\Gamma])$$

$$= \Gamma\text{-Hilb}(A^2)$$



$$[A^2/\Gamma]$$



$$Y_K$$

