Dogs talk: The Fano of lines, the Kuznetsov component, and a flop

[Kemboi – Segal] (Kimoi, Ed)

§0. Motivation

$Y \subseteq \mathbb{P}^5$ cubic 4-fold $\rightsquigarrow$ F_Y Fano of lines, sm. hyperkähler 4-fold

$\rightsquigarrow$ $\mathcal{A}_Y$ "non comm. K3-surface"

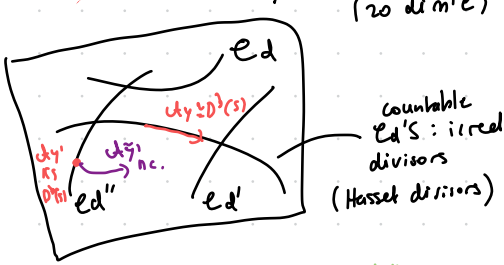
If A_Y is geometric, i.e. $\exists$ K3 S s.th. $\mathcal{A}_Y \simeq D^b(S)$, we get

$$D^b(S^{[2]}) \overset{[BKR]}{\simeq} D^b_{\mathfrak{A}_2}(S \times S) \simeq \mathrm{Sym}^2 \mathcal{A}_Y$$

Theorem (A): For any smooth cubic 4-fold Y, we have an equivalence

$$D^b(F_Y) \simeq \mathrm{Sym}^2 \mathcal{A}_Y$$

Vague motivation:



- on $\mathcal{C}_d$'s : $\mathcal{A}_Y$ is geometric
  i.e. $\mathcal{A}_Y \simeq D^b(S)$
  $\rightsquigarrow$ deforming $\mathcal{A}_Y$ within $\mathcal{C}_d$ $\rightsquigarrow$ stays geometric
- If you deform $\mathcal{A}_Y$ away from $\mathcal{C}_d$: $\mathcal{A}_Y$ will become non-comm.
  but: $\mathrm{Sym}^2 \mathcal{A}_Y$ still makes sense.
- Also: moduli of pol. HK 4-folds (def eq $S^{[2]}$) and containing F_Y is 20-dim'l

remember: $\mathcal{C}_d$'s contain a specific lattice $K \subseteq H^{2,2}(Y) \cap H^4(Y, \mathbb{Z})$ that is a K3 lattice, d is disc. of that. $d = h^2$ ← polarit. of K3 surface (basically rank wrt the polariz. h.)

Question (Galkin): Is $\mathrm{Sym}^2 \mathcal{A}_Y$ geometric? In plc. $\mathrm{Sym}^2(\mathcal{A}_Y) \simeq D^b(F_Y)$?

§1. Basics on GIT

Let G = reductive group (over $\mathbb{C}$ this means: every fd. representation is a direct sum of irreducibles), $G \curvearrowright V$ linearly on v-sp V; let $\chi: G \to \mathbb{C}^\times$ be a character. Then def $G \curvearrowright V \times \mathbb{C}_z$ by: $g \cdot (v, z) = (g \cdot v, \chi(g)^{-1} z)$ (linearization)

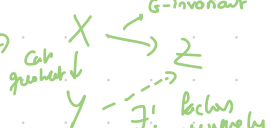
Def (topological) Let $z \neq 0$.

V is (χ-) semistable $\Leftrightarrow$ $\overline{G \cdot (v, z)} \cap (V \times \{0\}) = \emptyset$

V is (χ-) stable $\Leftrightarrow$ $G \cdot (v, z)$ is closed in $V \times \mathbb{C}$ and G_v is finite

Def (GIT quotient) GIT quotient = good quotient of semistable locus: (G-invar, surj., fcts on $V^{ss}/\!/_\chi G$ are G-invariant fct on V^{ss}...)

$$\varphi: V_\chi^{ss} \longrightarrow V/\!/_\chi G$$



it exists + is unique (bc. good quotients are categorical). (G reductive)

Properties:

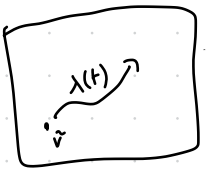
- pts in $V/\!/_\chi G$ corresp. to clsd orbits in V_χ^{ss};
- φ identifies points if their orbit closures intersect in V_χ^{ss};
- On V_χ^s φ is a honest orbit space;
- If $V_\chi^{ss} = V_\chi^s$, then $V/\!/_\chi G = V_\chi^{ss}/G$ or $[V_\chi^{ss}/G]$ as stack. (just quotient with taking invariant fcts)

$V/\!/_\chi G$ is... stacle orbifold variety

V^{ss} V^s V^s + trivial stabilizers

Def (1-PS) A 1-parameter subgroup = group hom $\mathbb{C}^\times \xrightarrow{\lambda} G$ (non-trivial).

Hilbert-Mumford Criterion $\tilde{v} = (v, 1)$ let $(*) := \lim_{t \to 0} \lambda(t) \cdot \tilde{v}$



v is unstable $\Leftrightarrow \exists \lambda$ sth.: $(*)$ exists and $\in V \times \{0\}$

v is semistable $\Leftrightarrow \forall \lambda$: If $(*)$ exists, then $\notin V \times \{0\}$

v is stable $\Leftrightarrow \forall \lambda$: $(*)$ does not exist

§2. Examples towards Kemboi-Segal's setting

1) Orbifold flop: Let $\mathbb{C}^{n+1} \circlearrowleft \mathbb{C}^\times$ by $t \cdot (\underline{x}, p) = (t \cdot \underline{x}, t^{-n} p)$ w\ weights $(1, \dots, 1, -n)$

- linearize: $\mathbb{C}^{n+1} \times \mathbb{C}_z \circlearrowleft \chi_k$, $\chi_k : \mathbb{C}^\times \to \mathbb{C}^\times$, $t \mapsto t^k$ on $\mathbb{C}_z$: $z \mapsto t^{-k} z$.
- 1-PS = characters: $\mathbb{C}^\times \xrightarrow{\lambda} \mathbb{C}^\times : t \mapsto t^r$
- for $\tilde{v} = (\underline{x}, p, 1)$ have: $\lambda(t) \cdot \tilde{v} = (t^r \underline{x}, t^{-nr} p, t^{-kr})$

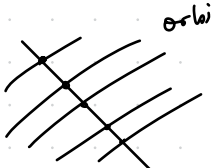
$k > 0$: land in $V \times \{0\}$ if $-kr > 0 \Rightarrow r < 0$ (2nd coordinate $= 0$)

$\Rightarrow$ need $\{\underline{x} \neq 0\}$ for (semi-)stability

get $V^{ss} = \mathbb{C}^n \setminus \{0\} \times \mathbb{C} \,/\!/_{\mathbb{C}^\times} \rightsquigarrow X_+ = V_+^{ss}/\mathbb{C}^\times = \mathrm{Tot}_{\mathbb{P}^{n-1}}(\mathcal{O}(-n))$

$k < 0$: Get $V^{ss} = V^s = \{p \neq 0\}$ claim: $X_- = V_-^{ss}/\mathbb{C}^\times = [\mathbb{C}^n/\mu_n]$

slice argument: $S := \{(\underline{x}, 1)\} \subseteq V \times \mathbb{C}$: • every orbit intersects S

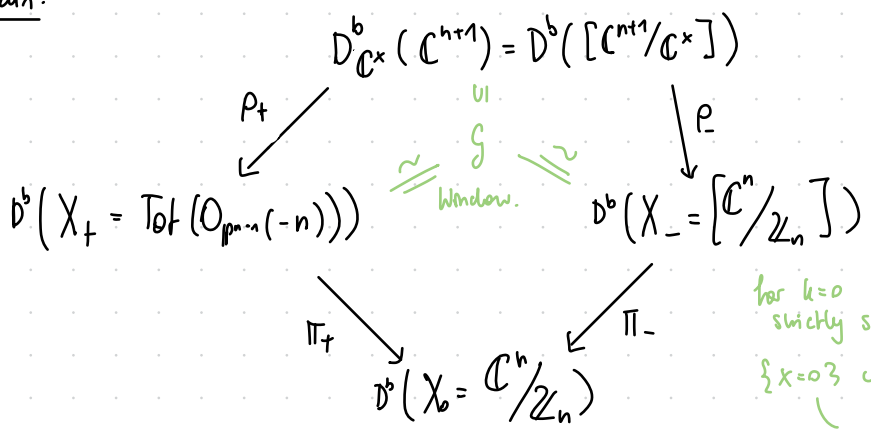


• what does remain of the $\mathbb{C}^\times$-action on S?

$\rightsquigarrow \mu_n \circlearrowright S$, since 1 needs to be fixed to stay in S! ✳

$k=0$: $V^{ss} = V$, but $V^s = \{p \neq 0 \wedge x \neq 0\}$

We obtain:



for $k=0$ we have strictly semistable pts $\{x=0\} \cup \{p=0\}$ collapsed into origin in X_0!

2) Grassmanian $Gr(2,6)$

Let $V = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6)$ $\circlearrowleft$ $GL_2(\mathbb{C})$ by $g \cdot A = Ag^{-1}$ (preserves $\mathrm{im} A \leq \mathbb{C}^6$)

- linearization: $V \times \mathbb{C} \circlearrowleft \chi_k$ and characters are $\chi_k : GL_2(\mathbb{C}) \to \mathbb{C}^*$, $A \mapsto \det(A)^k$
- $\{$ 1-PS of $GL_2\}$ $\overset{1:1}{\longleftrightarrow}$ $\{$ decoup. $\mathbb{C}^2 = \mathbb{C}f_1 \oplus \mathbb{C}f_2$ with weights $(a,b) \in \mathbb{Z}^2 \mid a \geq b\}$

$$\begin{bmatrix} \mathbb{C}^\times \to GL_2(\mathbb{C}) \\ t \mapsto \lambda(t) \end{bmatrix} \rightsquigarrow \text{in that basis } \lambda(t) = \mathrm{diag}(t^a, t^b)$$

Now for Hilbert-Mumford:

$$\chi_k(\lambda_{a,b}(t)) = t^{k(a+b)}$$

$$\lambda(t) \cdot (A, 1) = (t^{-a} A f_1, t^{-b} A f_2, t^{-k(a+b)})$$

↗ unstable, if λ exists: $a+b > 0$

$k<0$:
- $\mathrm{rk}\, A = 2$: $Af_1, Af_2 \neq 0$, therefore if $a+b>0 \Rightarrow a>0$ or $b>0$ $\Rightarrow$ 1st or 2nd compont $\to \infty$ $\Rightarrow$ limit does not exist for any λ $\Rightarrow$ A stable
- $\mathrm{rk}\, A \leq 1$: unstable $\Rightarrow V_-^{ss} = V_-^s = \{\mathrm{rk} A = 2\}$ $\Rightarrow X_- = V_-^s / GL_2 = Gr(2,6)$

(note: GL_2-action on $\mathrm{rk}(A) = 2$ is free!)

$k>0$: ϕ and $k=0$: pt.

§3. Window-setup Kemboi-Segal + proof sketch

Let $Y \subseteq \mathbb{P}^5$ sm. cubic hourfold, $F_Y = \{[L] \in Gr(2,6) \mid L \subseteq Y\}$

$V(f)$; $f \in H^0(\mathcal{O}(3))$

claim: $F_Y = V(s_f) \subseteq Gr(2,6)$, $s_f \in H^0(Gr(2,6), \mathrm{Sym}^3 \mathcal{U}^\vee)$ ← tautological bundle.

pf: We can define a section:

(W = line : $W = [t \cdot v_0 + s \cdot v_1]$, $L(t \cdot v_0 + s \cdot v_1) \in \mathrm{Sym}^3 W^\vee$ ✓)

$$\begin{array}{ccc} \mathrm{Sym}^3 \mathcal{U}^\vee & \ni & \mathrm{Sym}^3 \mathcal{U}^\vee \\ \text{rk } 4 \downarrow \pi & & \downarrow \\ Gr(2,6) & \ni & [U] \end{array} \quad \rightsquigarrow \text{def: } s_f : [U] \mapsto f|_U \in \mathrm{Sym}^3 U^\vee$$

$$\Rightarrow F_Y = \{[U] \in Gr(2,6) \mid s_f([U]) = 0\} = V(s_f) \subseteq Gr(2,6) \#$$

(Actually: s_f is transverse, i.e. $\forall x \in V(s_f)$: $d s_{f,x} : T_x Gr(2,6) \to \mathrm{Sym}^3(\mathcal{U})_x$ is surjective.

This is relevant for $\mathrm{Crit}(W) = V(s_f)$ later)

KP:

$$D^b(F_Y) \cong MF(\mathrm{Tot}_{Gr(2,6)}(\mathrm{Sym}^3 \mathcal{U}^\vee), W([U], \Psi))$$

(Ψ viewed in $\mathrm{Sym}^3 \mathbb{C}^6$ instead of $\mathrm{Sym}^3 U$)

$$\langle f|_W, \Psi \rangle = f(\mathrm{Sym}^3 i(\Psi))$$

$i : U \hookrightarrow \mathbb{C}^6$

Now as a GIT quotient:

$$V = \underbrace{\mathrm{Hom}(U, \mathbb{C}^6)}_{16\text{-dim'l}} \oplus \mathrm{Sym}^3 U \ni (\Xi, \Psi) \circlearrowleft G = GL(U) \cong Gl_2$$

by the action: $g \cdot (\Xi, \Psi) = (\Xi \cdot g^{-1}, \mathrm{Sym}^3 g(\Psi))$

means applying g to the basis of U and expanding Ψ

$\Psi = \Psi_{30} u^3 + \Psi_{21} u^2 v + \Psi_{12} u v^2 + \Psi_{03} v^3$

for $U = \langle u, v \rangle$.

Hilbert-Mumford Criterion gives:

$k<0$: $V_-^{ss} = V_-^{s} = \{\mathrm{rk}(\Phi) = 2\} \times \mathrm{Sym}^3 \mathcal{U}$ (no cond on Ψ!)

and one verifies that: $q: V_-^{ss} \to \mathrm{Tot}(\mathrm{Sym}^3 \mathcal{U})$ is a principal Gl_2-bundle

$(\Phi, \Psi) \mapsto (\lceil \mathrm{im}\Phi \rceil, \mathrm{Sym}^3 \Phi(\Psi))$

$\rightsquigarrow X_- = V_-^{ss}/Gl_2 \xrightarrow{\sim} \mathrm{Tot}(\mathrm{Sym}^3 \mathcal{U})$.

for $k>0$

semistability if (actually $\Leftrightarrow$ stability)
- $\Psi \neq 0$ and has at least two distinct roots
- $\mathrm{im}(\Phi)$ is not contained in double root of Ψ.

Note: we have a stabilization: $\{0\} = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 = \mathrm{Sym}^3 \mathcal{U}^\vee$ (Σ_1: $*$ root)

$\rightsquigarrow$ get orbifold $X_+ = [V_+^{ss}/Gl_2]$, also how to see that:

Let $X_+^\circ \subseteq X_+$ be the open substack where Ψ has distinct roots

$Gl_2 \curvearrowright \Sigma_3$ transitively $\Rightarrow$ take slice $\mathrm{Hom}(\mathcal{U}, \mathbb{C}^6) \times \{\Psi_0\}$, then fix $\Psi_0 = t_1^3 + t_2^3$

stabilizer is $\Gamma = \mathrm{Stab}((0, \Psi_0)) = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

(analogous to the slice argument of $[\mathbb{C}^n/\mathbb{Z}_n]$) $\Rightarrow X_+^\circ = [\mathbb{C}^6 \times \mathbb{C}^6/\Gamma]$

Now $D = X_+ \setminus X_+^\circ$ divisor and one can check $D = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 6})$.

$\rightsquigarrow$ no semistable points with infinite stabilizer $\#$

Now connecting it with $\mathrm{Sym}^2 \mathcal{A}_Y$:

For $V(f) = Y = \mathbb{P}^5$ we get by Orlov's thm (actually KP + orbifold flop w/ different weights): $\mathcal{A}_Y \cong MF([\mathbb{C}^6/\mathbb{Z}_3], f)$

and then: $\mathrm{Sym}^2 \mathcal{U}_y \cong MF([\mathbb{C}^6 \times \mathbb{C}^6/\Gamma], f_1 + f_2)$

($f_1 + f_2$: cubic on each of the factors)

So we have:

$$\begin{array}{ccccc} & & \mathcal{H} = [V/Gl_2] & & \\ & \swarrow^{p_-} & \uparrow \mathcal{U} & \searrow^{p_+} & \\ X_- = \mathrm{Tot}(\mathrm{Sym}^3 \mathcal{U}) & & G & & X_+ \cong X_+^\circ \cong \mathrm{Sym}^2 \mathcal{U}_y \\ \downarrow & & & & \\ MF(X_-, W) \cong D^b(F_y) & & & & \end{array}$$

Brais: Will explain how we get the equivalence $D^b(X_-) \cong D^b(X_+)$ via window categories.