# DEMYSTIFYING BRIDGELAND STABILITY

# I: Why care?

① Algebra — Geometry

$D^b(\text{mod}A)$ — $D^b(\text{Coh}X)$

Triangulated Category

compare?

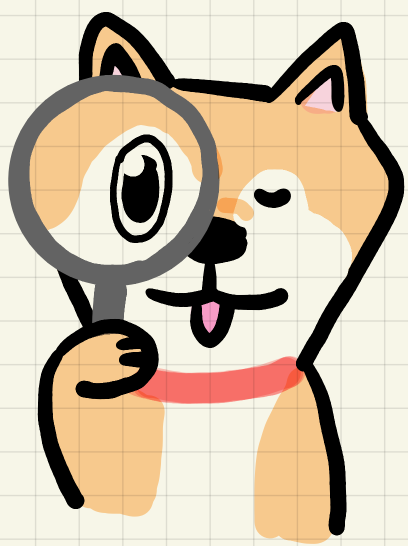
$\text{Stab}(-)$

$\boxed{\text{Complex Manifold}}$

Bridgeland: "Spaces of stability conditions" " "moduli of mirrors" "

[Bridgeland –Smith] Teichmüller spaces

[Bayer –Bridgeland] Autoequivalences



## ② Classification problems

vector bundles

modules over f.d. algebras (e.g. quiver reps)

**GOAL:** classify all objects of some kind

too many!

all objects built from "stable" ones

Fix discrete invariants $\underline{v}$ (e.g. rank, degree)

$\rightsquigarrow M^{st}(\underline{v})$ : moduli of stables of class $\underline{v}$.

**HOPE:** $M^{st}(\underline{v})$ is "nice"

e.g. smooth & projective:

slope ✓ Gieseker ✓

Bridgeland: some examples

could have different ways to be "stable"

**BRIDGELAND STABILITY BONUS**

deformation + wall crossing properties

$\sigma_1$ - - - - - $\sigma_2$

Stab($\mathcal{D}$)

$$M^{st}_{\sigma_1}(\underline{v}) \dashrightarrow M^{st}_{\sigma_2}(\underline{v})$$

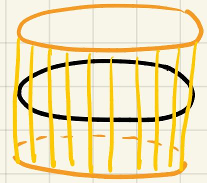
$\rightsquigarrow$ MMP / birationality

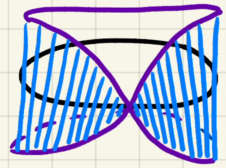
[Bayer-Macrì '14]

$\rightsquigarrow$ Brill-Noether loci

e.g. [Feyzbakhsh '20]

# II. Axiomatising Stability





## slope stability

C: sm. proj. curve / $\mathbb{C}$

E: ~~vector bundle~~ $\in \mathrm{Coh}(C)$ $\rightsquigarrow$ $\mu(E) = \begin{cases} +\infty & \mathrm{rk}=0 \\ \dfrac{\deg(E)}{\mathrm{rk}(E)} & \end{cases}$

**Def$^n$** E is $\mu$-(semi)stable if $0 \neq F \subsetneq E \Rightarrow \mu(F) \leq \mu(E)$

**e.g.** $\mathcal{L}$: line bundle $\mu(\mathcal{L}) = \deg(\mathcal{L})$

- $0 \neq F \subseteq \mathcal{L}$ must have rank 1

  but $\deg(F) < \deg(\mathcal{L})$ ☺ $\mathcal{L}$ is stable

**Th$^m$** E has a unique Harder-Narasimhan filtration

$0 = E_0 \subset E_1 \subset \dots E_{n-1} \subset E_n = E$ s.t.

(i) $E_i/E_{i-1}$ $\mu$-semistable (ii) $\mu(E_1/E_0) > \dots > \mu(E_n/E_{n-1})$

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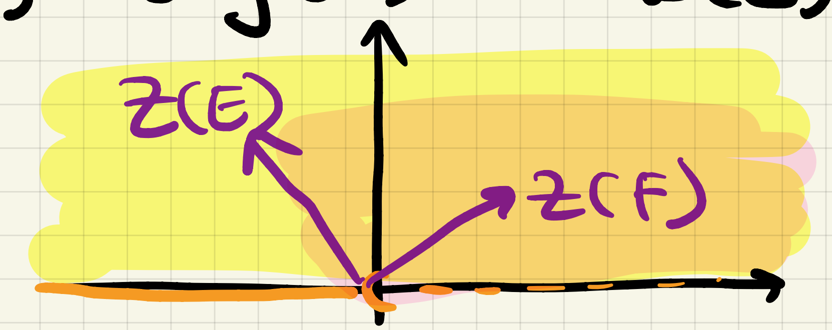
⚠ $Z_\mu(E) := -\deg(E) + i\,\mathrm{rank}(E) \in \mathbb{H}$ $\neq 0$

E stable,

$F \subset E$




$Z(E) = m e^{i\pi\phi}$

$\mathbb{C}$

# CHEAT SHEET: SLOPE STABILITY

① $E, F \in \mathrm{Coh}(C)$

② $E$ is semistable of phase $\phi$,

$$F \hookrightarrow E \Rightarrow \phi(F) \leq \phi(E)$$

i.e. $\nexists$ map $F' \to E$ s.t. $\phi(F) > \phi(E)$

③ $E$ is stable of phase $\phi$,

$$F \hookrightarrow E \Rightarrow \phi(F) < \phi(E)$$

i.e. $E$ has no nontrivial subobjects of the same phase

④ $E$ has a unique HN filtration by stables

$$\boxed{\begin{tikzcd} 0 = E_0 \arrow[r, hook] & E_1 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & E_{n-1} \arrow[r, hook] & E_n = E \arrow[d, two heads] \\ & E_1/E_0 & \cdots & & E_n/E_{n-1} \\ & \phi_1 > & \cdots & > & \phi_n \end{tikzcd}}$$

⑤ $Z_\mu(E) = -\deg(E) + i\,\mathrm{rank}(E) \in \mathbb{H}$

$$E \neq 0 \Rightarrow Z_\mu(E) = \underbrace{m(E)}_{\in \mathbb{R}_{>0}} \cdot e^{i\pi\underbrace{\phi}_{\phi \in (0,1]}}$$

# CHEAT SHEET: BRIDGELAND STABILITY

① $E, F \in \mathcal{D}$ : triangulated category

"semistable"

② $E \in P(\phi)$ : subcat. of $\mathcal{D}$ $\forall \phi \in \mathbb{R}$

$\phi_1 > \phi_2$, $F \in P(\phi_1)$, $E \in P(\phi_2)$

$\Rightarrow \mathrm{Hom}(F, E) = 0$

③ $E \in P(\phi)$ is **stable** if $E$ is a simple object of $P(\phi)$

③.5 $P(\phi)[1] = P(\phi + 1)$

- $[1] : \mathcal{D} \to \mathcal{D}$ "shift"
- $A \to B \to C \to A[1]$ exact triangle

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd}$$

④ $E$ has a unique HN filtration by stables

$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\ A_1 \in P(\phi_1) \arrow[u, dashed] & \cdots & & A_n \in P(\phi_n) \arrow[u, dashed] & \end{tikzcd}$$

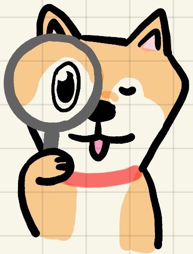
$\phi_1 > \cdots > \phi_n$

⑤ $Z : \underline{K(\mathcal{D})} \to \mathbb{C}$ group homomorphism

Grothendieck group $= \dfrac{(\text{free group generated by ob } \mathcal{D})}{[B] = [A] + [C] \Leftrightarrow A \to B \to C \to A[1]}$

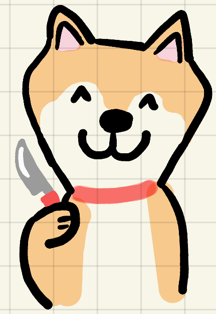
$$0 \neq E \in P(\phi) \Rightarrow Z(\underbrace{[E]}_{\in K(\mathcal{D})}) = \underbrace{m(E)}_{\in \mathbb{R}_{>0}} \cdot e^{i\pi\phi} \in \mathbb{C}$$

**PLUS** $\left.\begin{array}{l}\cdot\ Z \text{ factors via } \Lambda \simeq \mathbb{Z}^r \\ \cdot\ \text{support property}\end{array}\right\} (*)$



slicings ↓ ← central charge

**Def$^{\underline{n}}$** $\sigma = (P, Z)$ is a Bridgeland stability condition on $\mathcal{D}$



$\mathrm{Stab}_\Lambda(\mathcal{D})$ = all stability conditions on $\mathcal{D}$ wrt $\Lambda$

**Th$^{\underline{m}}$** [Bridgeland '07] $\mathrm{Stab}_\Lambda(\mathcal{D})$ is a $\mathbb{C}$-mfld.

$$\mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \simeq \mathbb{C}^r$$
$$\sigma = (A, Z) \longmapsto Z$$

↑ local homeomorphism

**Th$^{\underline{m}}$** [B '07] $\sigma = (P, Z)$ is equivalent to the data:

- $A$ is a ♡ of a bdd t-structure on $\mathcal{D}$
- $Z_A : K(A)\setminus\{0\} \to \mathbb{H}$ $\rightsquigarrow$ $\mu_{Z_A} = \dfrac{-\mathrm{Re}\, Z_A}{\mathrm{Im}\, Z_A}$

\+ HN property $\quad 0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_n = E$, $\quad \mu_{Z_A}(E_1/E_0) > \dots$

\+ $(*)$

<u>IDEA</u>: $\cdot\ (P, Z) \rightsquigarrow A = P(0,1] := \langle P(\phi) \rangle_{ext}^{\phi \in (0,1]}$

$\cdot\ (A, Z_A)$ $\cdot\ E \neq 0,\ Z_A(E) = m e^{i\pi\phi} \Rightarrow E \in P(\phi)$ $\quad \phi \in (0,1]$

$\cdot\ P(\phi') = P(n+\phi) = P(\phi)[n]$ $\quad \phi \in (0,1]$

# EXAMPLES:

## ① $\mathcal{D} = D^b X$

$\dim X = 1$, $g(X) \geq 1$: $\mathrm{Stab}(X) = (\mathrm{Coh}\,X, Z_\mu)\cdot \widetilde{GL_2^+(\mathbb{R})}$
$\simeq \mathbb{C} \times \mathbb{H}$

$X = \mathbb{P}^1$ : $D^b \mathbb{P}^1 \xrightarrow{\sim} D^b \underbrace{\mathrm{Rep}(\overbrace{\bullet \rightrightarrows \bullet}^{K_2})}$

$E^\bullet \longmapsto \mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1), E^\bullet)$

$\rightsquigarrow \sigma = (\mathrm{Rep}\,Q, Z')$

$\rightsquigarrow \mathrm{Stab}(X) \simeq \mathbb{C}^2$

$\dim X = 2$ $(\mathrm{Coh}\,X, Z)$ never works

<u>IDEA</u> torsion pair $(\mathcal{T}_{H,\beta}, \mathcal{F}_{H,\beta}) \overset{\text{tilt}}{\rightsquigarrow} \mathrm{Coh}^{H,\beta}(X)$

H-slope $< \beta$ H-slope $\geq \beta$ $= \langle \mathcal{F}[-1], \mathcal{T} \rangle_{\mathrm{ext}}$

$\sigma = (\mathrm{Coh}^{H,\beta}(X), \ldots)$

$\dim X = 3$ "double tilt sometimes works"

# EXAMPLES:

(2) $Q$: finite quiver, $Q_0 = \{0, \dots, n\}$

Consider $\mathrm{Rep}_{\mathbb{C}} Q$

(A) $K_2$: $0 \underset{y}{\overset{x}{\rightrightarrows}} 1$

$\underline{V} \in \mathrm{Rep}_{\mathbb{C}} K_2$ : $(V_0, V_1, \phi_x, \phi_y : V_0 \to V_1)$

e.g. $\mathbb{C} \underset{\mu \in \mathbb{C}}{\overset{\lambda \in \mathbb{C}}{\rightrightarrows}} \mathbb{C}$ $(1,1)$ $\qquad \mathbb{C}^n \underset{\phi_y}{\overset{\phi_x}{\rightrightarrows}} \mathbb{C}^m$

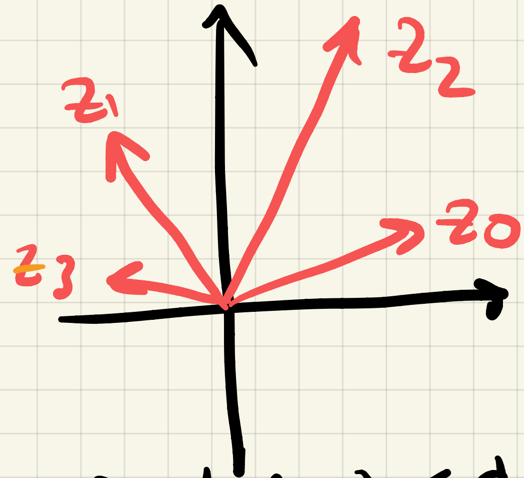
⚠ $d(\underline{V}) = (\dim V_0, \dim V_1)$

$(1,1)$: $(\lambda, \mu) + (0,0)$ – $\mathbb{C}\mathbb{P}^1$ family of modules w/ class $(1,1)$

$\mathbb{C} \underset{0}{\overset{0}{\rightrightarrows}} \mathbb{C} \rightsquigarrow M(1,1)$ is 1-dim variety + 0-dim.

want better behaved moduli $\rightsquigarrow$ use stability conditions

Pick $z_0, \dots, z_n \in \mathbb{H}$




$$Z(\underline{V}) = \sum_{i=0}^{n} \dim V_i \cdot z_i$$
$$= r e^{i\pi\phi}$$

$\underline{V}$ is stable if $\underline{W} \subsetneq \underline{V} \Rightarrow \phi(\underline{W}) < \phi(\underline{V})$

Ⓐ : Pick $z_0, z_1 \in \mathbb{H}$

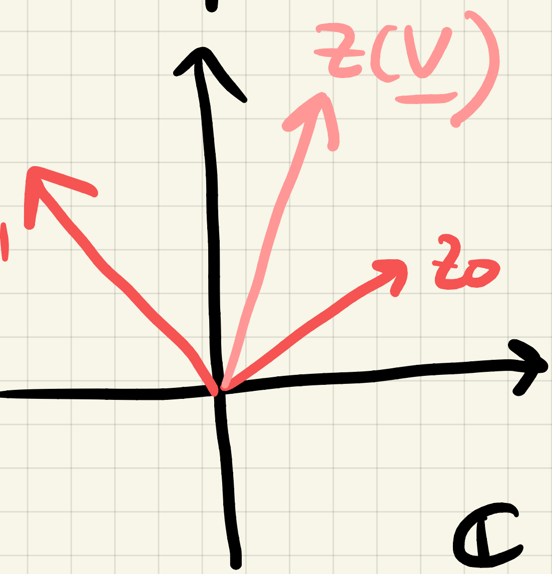
$$\begin{tikzcd}
S_1 & 0 \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "\alpha"] \\
\underline{V} & \mathbb{C}^n \arrow[r, shift left, "d_x"] \arrow[r, shift right, "d_y"'] & \mathbb{C}^m
\end{tikzcd}$$

simple rep.

If $m > 0$, $S_0$ is always a subrep.

Assume $\phi(z_1) > \phi(z_0)$

$Z(S_1) = z_1$




$n > 0 \Rightarrow \phi(S_1) > \phi(\underline{V})$

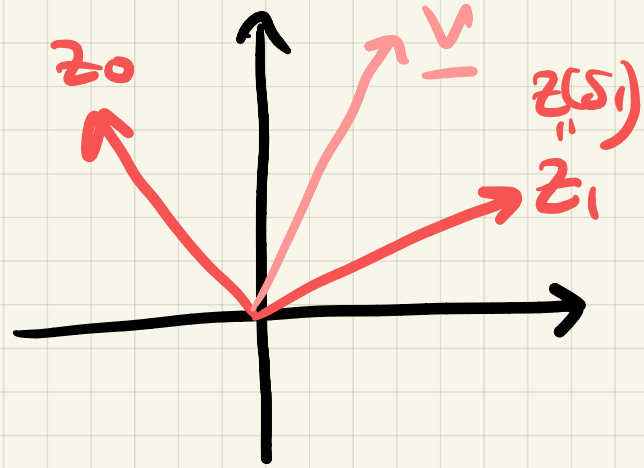
$\Rightarrow \underline{V}$ not stable

sim. argument for $S_0$: $\mathbb{C} \underset{0}{\overset{0}{\rightrightarrows}} 0$

$\rightsquigarrow$ $S_0, S_1$ only stable reps.

Assume $\phi(z_1) < \phi(z_0)$

consider




$$\begin{tikzcd} S_1 & 0 \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "\alpha"] \\ \underline{V}: & \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd}$$

$\phi(\underline{S_1}) < \phi(\underline{V})$ ✓ $(\underline{S_1} \subseteq \underline{V})$

**AND**

$$\begin{tikzcd} \underline{S_0} & \mathbb{C} \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d] & 0 \arrow[d, "0"] \\ \underline{V} & \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd}$$

$S_0 \subseteq \underline{V}$

$\Leftrightarrow$ this commutes

$\Leftrightarrow$ $(\lambda,\mu) = (0,0)$

$\phi(S_0) > \phi(\underline{V})$

$(\lambda,\mu) \neq (0,0) \Rightarrow \underline{V}$ stable

⚠ These corresp. to $GL_2^+(\mathbb{R}) \cdot \sigma_\mu$.