The special McKay correspondence and homological mirror symmetry for orbifold surfaces

Bogdan Simeonov

Overview

- Classical McKay correspondence for $SL_2(\mathbb{C})$
- Generalization to $GL_2(\mathbb{C})$ and reflexive modules on quotient surface singularities (Esnault and Wunram)
- Derived categories of orbifold surfaces and full exceptional collections (Ishii and Ueda)
- Homological mirror symmetry for orbifold log Calabi-Yau surfaces

The SL_2 McKay correspondence

Given a finite subgroup $G \subset SL_2(\mathbb{C})$, there is a bijection between non-trivial irreducible representations of G and exceptional curves in the minimal resolution of $X = \mathbb{C}^2/G$.

$$\text{Irreducible representations of } G \longleftrightarrow \text{Exceptional curves in } Y$$

Explicitly, to each (indecomposable reflexive) $\mathcal{O}_X$-module M, Verdier and Gonzalez-Sprinberg defined the torsion-free pullback $\pi^* M/\mathrm{torsion}$, the full sheaf associated to M. They showed that $c_1(\pi^* M/\mathrm{torsion})$ is Poincare dual to a divisor intersecting exactly one of the exceptional curves once. This Chern class completely determines the indecomposable reflexive module.

A derived equivalence

It was later noticed by Kapranov and Vasserot (and also Bridgeland-King-Reid for finite subgroups of $SL_3(\mathbb{C})$) that the McKay correspondence has an incarnation as a derived equivalence

$$D^b(\mathcal{X}) \simeq D^b(Y)$$

This derived equivalence is induced by a Fourier-Mukai functor associated to the correspondence given by a universal G-Hilbert scheme.

The GL_2 McKay correspondence

Origins: the work of Wunram and Esnault

Given a $\mathcal{O}_X$-module M, one can associate an $\mathcal{O}_Y$-module $\mathcal{M} := \pi^* M/\mathrm{torsion}$ - the resulting sheaf is called *full*.

Proposition (Esnault): *A sheaf $\mathcal{M}$ on Y is isomorphic to $\pi^* M/\mathrm{torsion}$ for a reflexive sheaf M on X if and only if $\mathcal{M}$ is locally free, generated by global sections and $H^1(Y, \mathcal{M}^\vee \otimes \omega_Y) = 0$. There is a bijective correspondence between full sheaves on Y and reflexive sheaves on X.*

In the case at hand, X is a normal surface and reflexive modules are the same as Cohen-Macaulay modules.

If $G \subset GL_2(\mathbb{C})$ there can be less exceptional curves in $Y = \widetilde{\mathbb{C}^2/G}$ than there are irreducible representations. To remedy this, Wunram defined the notion of special sheaves.

Theorem (Wunram): *A full sheaf $\mathcal{M}$ is defined to be special if $H^1(Y, \mathcal{M}^\vee) = 0$. There is a bijective correspondence between the exceptional curves $E \subset Y$ and the irreducible representations ρ of G for which $\mathcal{M}_\rho := \pi^*(\mathcal{O} \otimes \rho^*)^G/\mathrm{torsion}$ is a special full sheaf. The special $\mathcal{M}_\rho$ have a Chern class $c_1(\mathcal{M}_\rho)$ which is Poincare dual to a cycle intersecting exactly one E_i transversely at a point.*

The case of cyclic quotient surface singularities

When G is the finite cyclic subgroup of $GL_2(\mathbb{C})$ given by $\{\begin{pmatrix} \chi & 0 \\ 0 & \chi^q \end{pmatrix} \mid \chi^n = 1\}$ with $(n, q) = 1$, the resulting $\mathbb{C}^2/\frac{1}{n}(1, q)$ singularity has a resolution Y which can be understood by the Hirzebruch-Jung continued fraction expansion

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} := [b_1, \dots, b_r]$$

Wunram proved that the weights of the special representations form the I-series defined by the recursion

$$i_0 = n, \quad \underbrace{i_1 = q, \quad i_2 = b_1 i_1 - i_0, \quad \dots, \quad i_r = 1}_{\text{correspond to exceptional curves in } Y}, \quad i_{r+1} = 0$$

Derived categories perspective

Analogously to the case of $SL_2(\mathbb{C})$, one can define a correspondence

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\pi_{\mathcal{X}}"] \arrow[d, "\pi_Y"'] & \mathcal{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

which results in a fully faithful functor

$$\Phi : D^b(Y) \to D^b(\mathcal{X})$$
$$\Phi(\mathcal{F}) := \pi_{\mathcal{X}*}\pi_Y^*(\mathcal{F})$$
$$\mathcal{M}_\rho^\vee \mapsto \mathcal{O} \otimes \rho$$

When $X = \mathbb{C}^2/\frac{1}{n}(1, q)$, $D^b(Y)$ is generated by the indecomposable special full sheaves, by results of Van den Bergh, and hence the essential image of Φ is generated by $\mathcal{O} \otimes \rho$ where ρ is a special representation.

Since

$$\mathrm{Ext}^\bullet(\mathcal{O} \otimes \rho, \mathcal{O}_0 \otimes \tau) = \begin{cases} \mathbb{C}, & \tau = \rho \\ 0 \end{cases}$$

there is a semiorthogonal decomposition

$$D^b([\mathbb{C}^2/G]) = \langle \mathbf{e}, \Phi D^b(Y) \rangle$$

with $\mathbf{e}$ generated by the equivariant skyscraper sheaves $\mathcal{O}_0 \otimes \tau$ with τ a non-special representation.

The sheaves $\mathcal{O}_0 \otimes \tau$ do not always form an exceptional collection, however. For example, when $X = \mathbb{C}^2/\frac{1}{8}(1,3)$ there is both a morphism in $\mathrm{Ext}^2(\mathcal{O}_0 \otimes \rho_4, \mathcal{O}_0 \otimes \rho_2) = \mathbb{C}$ and one in $\mathrm{Ext}^2(\mathcal{O}_0 \otimes \rho_2, \mathcal{O}_0 \otimes \rho_6) = \mathbb{C}$. In general, the algebra of $\mathcal{O}_0 \otimes \tau$ is governed by the McKay quiver.

The McKay quiver

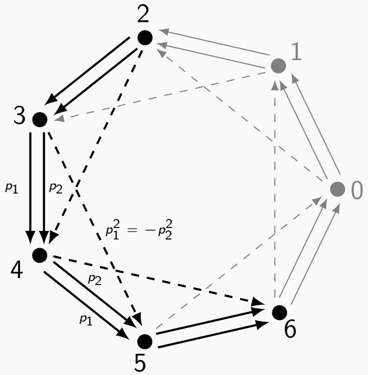

Figure 1: The case $\frac{1}{7}(1,1)$. Thick arrows are degree 1 and dashed ones are degree 2.

A full exceptional collection

While the equivariant skyscraper sheaves $\mathcal{O}_0 \otimes \tau$ might fail to produce an exceptional collection for the component **e**, there is an alternative collection of sheaves that is exceptional:

Theorem (Ishii-Ueda): *There are sheaves N_i indexed by the non-special representations and supported at the origin, together with a semiorthogonal decomposition*

$$D^b([\mathbb{C}^2/G]) = \langle N_1, \ldots, N_d, \Phi D^b(Y) \rangle$$

The sheaves N_d are defined as follows: if $i_{t-1} > d > i_t$, then N_d is the sheaf associated to the module

$$\mathbb{C}[x, y]/(x, y^{j_t}) \otimes \rho_{d+q-j_t q}$$

where j_t is the dual to the i_t series satisfying $q^{-1} i_t \equiv j_t \mod n$.

The derived category of an orbifold surface

While the previous theorem was stated in the local case $\mathbb{C}^2/G$, it can be extended to arbitrary surfaces with cyclic quotient singularities.

Theorem (Ishii-Ueda): *Let $\mathcal{X}$ be a surface with cyclic quotient singularities and Y the minimal resolution of the coarse space X of $\mathcal{X}$. Then there is a semiorthogonal decomposition*

$$D^b(\mathcal{X}) = \langle N_{p_1,1}, \ldots, N_{p_1,d_1}, N_{p_2,1} \ldots, N_{p_s,d_s} \Phi D^b(Y) \rangle$$

where the $N_{p_i,j}$ are sheaves supported at the orbifold points p_i of $\mathcal{X}$.

As a corollary, we obtain that $D^b(\mathcal{X})$ admits a full exceptional collection whenever $D^b(Y)$ does.

A left adjoint

It is possible to describe the gluing bimodule between $\mathbf{e}$ and $\Phi D^b(Y)$ via the left adjoint Ψ to Φ.

Theorem (Ishii, Rota-Gugiatti): *The left adjoint Ψ to Φ satisfies*

$$\Psi(\mathcal{O}_0 \otimes \rho_i) = \begin{cases} \mathcal{O}_F(F)[1], & \textit{if } i+q+1 \equiv 0 \mod n \\ \mathcal{O}_{E_t}(-b_t+1) & \textit{if } i+q+1 \equiv i_t \textit{ is special} \\ 0 & \textit{otherwise} \end{cases}$$

Example

Suppose that $\mathcal{X}$ has a singularity of type $\frac{1}{k}(1,1)$. Then the minimal resolution of X has a single exceptional $-k$ curve E, which is the whole fundamental cycle F of the resolution. One has

$$\Psi(\mathcal{O}_0 \otimes \rho_{k-2}) = \mathcal{O}_E(-k)[1], \quad \Psi(\mathcal{O}_0 \otimes \rho_{k-1}) = \mathcal{O}_E(-k+1)$$

Homological mirror symmetry for orbifold log Calabi-Yau surfaces

Homological mirror symmetry for log Calabi-Yau surfaces

Suppose (Y, D) is a pair of a compact projective surface Y and an anticanonical cycle of rational curves D with n nodes.

Theorem (Hacking-Keating): *Associated to a log CY pair (Y, D), there is an exact Lefschetz fibration $w : W \to \mathbb{C}$ whose general fiber is an n-punctured torus T^0. When (Y, D) is at the large complex structure limit, there is an equivalence*

$$D^b(Y) \simeq D^b\mathcal{F}uk^{\to}(w)$$

The construction of w uses the following strategy: one takes a full exceptional collection of $D^b(Y)$ and restricts it to $\mathrm{Perf}(D)$ which is mirror to $\mathrm{Fuk}(T^0)$ where T^0 is a punctured torus. The images of the exceptional sheaves under this define Lagrangian S^1s inside T^0. One then creates a Lefschetz fibration $w : W \to \mathbb{C}$ whose general fiber is T^0 admitting those S^1s as vanishing cycles.

In other words, given $\mathcal{L} \in D^b(Y)$, one gets a line bundle $\mathcal{L}|_D \in \mathrm{Perf}(D)$ which is mirror (by work of Polishchuk-Lekili) to a Lagrangian circle in T^0, which is then used to define a vanishing cycle on an abstract Lefschetz fibration.

$$\begin{tikzcd}
D^b(Y) \arrow[r, dotted] \arrow[d] & D^b\mathcal{F}uk^{\rightarrow}(w) \arrow[d] \\
\mathrm{Perf}(D) \arrow[r, "\simeq"'] & \mathrm{Fuk}(T^0)
\end{tikzcd}$$

Extension to log Calabi-Yau orbifolds

Suppose $(\mathcal{X}, D^{orb})$ is obtained by taking a smooth log CY pair (Y, D) and contracting disjoint chains of rational curves in D. One has, as before, the decomposition

$$D^b(\mathcal{X}) = \langle \mathbf{e}, \Phi D^b(Y) \rangle$$

The first step in proving mirror symmetry for $(\mathcal{X}, D^{orb})$ is to understand the mirror to D^{orb}.

The work of Lekili-Polishchuk

Theorem (Lekili-Polishchuk): *There is an equivalence of categories*

$$\mathrm{Perf}(D^{orb}) \simeq \mathrm{Fuk}(\Sigma^0)$$

where Σ^0 is obtained as follows: for each rational curve component D_i of D^{orb}, there is an associated cylinder C_i. If two components D_i and D_{i+1} meet at a $\frac{1}{n}(1,q)$ point, one attaches n handles gluing ∂C_i^+ to ∂C_{i+1}^-, so as to identify the points $\frac{a}{n} \in \mathbb{R}/\mathbb{Z}$ with $-\frac{\overline{q^{-1}}a}{n} \in \mathbb{R}/\mathbb{Z}$.

An important observation is the following: the map $a \mapsto -\overline{q^{-1}}a \mod n$ is order-preserving on the subset $\{i_0, i_1, \ldots, i_r\} \subset \{0, 1, \ldots, n-1\}$. It implies that if one only attaches handles associated to the special I-series i_t for each nodal point of D^{orb}, the resulting Riemann surface will be a torus $T^0 \subset \Sigma^0$.

This suggests there is a commutative diagram

$$\begin{tikzcd}
D^b(Y) \arrow[r, "\Phi"] \arrow[d] & D^b(\mathcal{X}) \arrow[d] \\
\mathrm{Perf}(D) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Perf}(D^{orb}) \arrow[d, "\simeq"] \\
\mathrm{Fuk}(T^0) \arrow[r] & \mathrm{Fuk}(\Sigma^0)
\end{tikzcd}$$

where the bottom map is the fully faithful functor induced by the inclusion $T^0 \subset \Sigma^0$.

As a first step in the construction of the mirror to $(\mathcal{X}, D^{orb})$, one takes a Lefschetz fibration with general fiber Σ^0 and vanishing cycles the same as the ones in the Hacking-Keating mirror, under the inclusion $T^0 \subset \Sigma^0$. One then needs to construct the mirrors to the sheaves N_i.

Equivariant Koszul resolutions

The sheaf N_d has (locally, not necessarily globally) a resolution by projective modules

$$[\mathcal{O} \otimes \rho_{1+d+q} \to \mathcal{O} \otimes \rho_{1+d+q-j_t q} \oplus \mathcal{O} \otimes \rho_{d+q} \to \mathcal{O} \otimes \rho_{d+q-j_t q}] \simeq N_d$$

Motivated by this, one can construct mirror Lagrangian circles:

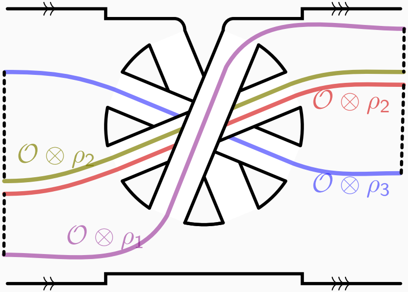

Figure 2: The mirror to a $\frac{1}{n}(1,1)$ point is given by two cylinders glued together with n handles. The picture above depicts the Lagrangian $\tilde{L}_1$ mirror to the Koszul resolution of $\mathcal{O}_0 \otimes \rho_1$ in the case $n = 5$.

Another example: a surface with a $\frac{1}{5}(1,3)$ point

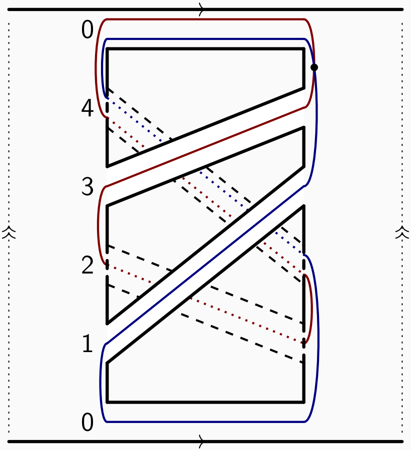

Figure 3: The general fiber of the Lefschetz fibration associated to a surface with a $\frac{1}{5}(1,3)$ orbifold point. The non-special handles are drawn with a dash. The blue curve describes the Lagrangian $\tilde{L}_2$ and the red one describes $\tilde{L}_4$.

Homological mirror symmetry for orbifold log CY surfaces

Theorem (S.): *Suppose $(\mathcal{X}, D^{orb})$ is an orbifold pair obtained from a smooth log CY pair (Y, D) at large complex structure limit by contracting a disjoint set of curves $E_i \subset D$ with $E_i \cdot E_i = -k_i < -2$. Then there is a commutative diagram*

$$\begin{array}{ccc} D^b(Y) & \xrightarrow{\simeq} & D^b\mathcal{F}uk^{\rightarrow}(w) \\ \Phi \big\downarrow & & \big\downarrow \\ D^b(\mathcal{X}) & \xrightarrow[\simeq]{} & D^b\mathcal{F}uk^{\rightarrow}(w') \end{array}$$

intertwining the fully faithful embedding Φ defined via a Fourier-Mukai kernel by Ishii-Ueda and the fully faithful inclusion induced by Lefschetz stabilization.

Work in progress: extension to all cyclic quotient singularities.

Example

Consider Y to be the blowup of $\mathbb{P}^2$ at $k+1$ points on a line (with k odd), and D the strict transform of the toric boundary of $\mathbb{P}^2$. Contracting the $-k$ line where the blowups occured produces an orbifold surface $(\mathcal{X}, D^{orb})$ with a single $\frac{1}{k}(1,1)$ point, in fact this is a hypersurface of degree $k+1$ in $\mathbb{P}(1,1,1,k)$. Notice that $U = Y \setminus D = \mathcal{X} \setminus D^{orb}$ is an open log CY variety.

The mirror to (Y, D) is a space $U^\vee$, equipped with a potential function $w : U^\vee \to \mathbb{C}$ whose general fiber is a torus with three punctures. The mirror to $(\mathcal{X}, D^{orb})$ is the same space $U^\vee$ equipped with a different function w' whose general fiber is of genus $\frac{k+1}{2}$ and has two boundary components. The two LG models differ by a process called Lefschetz stabilization. This is illustrated in the next slide.

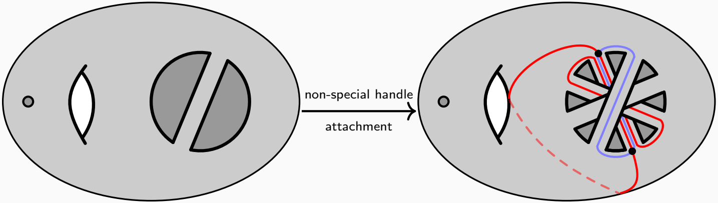

Figure 4: The case $k = 5$: a surface of genus 3 with two boundary components, which can be viewed as a torus with three boundary components and $k - 2 = 3$ handles attached. The blue curve describes the vanishing cycle $\tilde{L}_{k-1}$ mirror to $\mathcal{O}_0 \otimes \rho_4$ and the red one describes the mirror $\tilde{L}_{k-2}$ to $\mathcal{O}_0 \otimes \rho_3$.

Thank you for your attention!

References

- Esnault, H. *Reflexive modules on quotient surface singularities*
- Wunram, J. *Reflexive modules on quotient surface singularities*
- Ishii, A. and Ueda, K. *The special McKay correspondence and exceptional collections*
- Hacking, P. and Keating, A. *Homological mirror symmetry for log Calabi-Yau surfaces*
- Lekili, Y. and Polishchuk, A. *Arithmetic mirror symmetry for genus 1 curves with n marked points* and *Auslander orders over nodal stacky curves and partially wrapped Fukaya categories*
- Gugiatti, G. and Rota, F. *Full exceptional collections for anticanonical log del Pezzo surfaces*