Weak stability conditions on coherent systems of genus four curves

Nicolás Vilches

Columbia University

March 24th, 2026

Stability conditions

Setup:

- $\mathscr{D}$ triangulated category,
- Λ finite dimensional lattice,
- $v \colon K(\mathscr{D}) \to \Lambda$ additive.

Key example: $\mathscr{D} = \mathrm{D}^b(C)$, $\Lambda = \mathbb{Z}^2$, $v(E) = (\deg E, \operatorname{rk} E)$

Stability conditions

Setup:

- $\mathscr{D}$ triangulated category,
- Λ finite dimensional lattice,
- $v\colon K(\mathscr{D}) \to \Lambda$ additive.

Key example: $\mathscr{D} = \mathrm{D}^b(C)$, $\Lambda = \mathbb{Z}^2$, $v(E) = (\deg E, \operatorname{rk} E)$

A **Bridgeland stability condition** on $\mathscr{D}$ is $\sigma = (Z, \mathcal{A})$, with:

- $Z\colon \Lambda \to \mathbb{C}$ *central charge*,
- $\mathcal{A} \subset \mathscr{D}$ *heart* of a *bounded t-structure*.

We impose:

- Z maps $\mathcal{A}$ to the upper half-space. We set $\mu(E) := -\operatorname{Re} Z(E)/\operatorname{Im} Z(E)$.
- Existence of Harder–Narasimhan filtrations.
- Support property.

We impose:

- Z maps $\mathcal{A}$ to the upper half-space. We set $\mu(E) := -\operatorname{Re} Z(E)/\operatorname{Im} Z(E)$.
- Existence of Harder–Narasimhan filtrations.
- Support property.

Example. Take $\mathscr{D} = \mathrm{D}^b(C)$, and set

$$\mathcal{A} = \mathrm{Coh}(C), \quad Z(E) = -\deg(E) + i\,\mathrm{rk}(E).$$

This recovers slope-stability on C.

Bridgeland's deformation theorem

Theorem (Bridgeland, 2007)

The set $\mathrm{Stab}(\mathscr{D})$ *carries a natural topology. Moreover, the forgetful map*

$$\mathcal{Z}\colon \mathrm{Stab}(\mathscr{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C}), \qquad \sigma = (Z, \mathcal{A}) \mapsto Z$$

is a local homeomorphism.

(Classical) coherent systems

Recall: A **coherent system** on C is a pair (E, V), where

- $E \in \mathrm{Coh}(C)$,
- $V \subset H^0(C, E)$.

Many interesting questions: non-emptiness of moduli spaces, higher rank Brill–Noether theory, etc...

Last year: study coherent systems using derived categories.

- Alexeev–Kuznetsov *Augmentations, reduced ideal point gluings and compact type degenerations of curves*,
- Feyzbakhsh–Novik *Derived category of coherent systems on curves and stability conditions*.

The category $\mathcal{T}_C$

Set

$$\mathcal{T}_C = \{(E, V; \phi) : E \in \mathrm{Coh}(C),\ \phi\colon V \to H^0(C, E)\}.$$

These are *almost* coherent systems; note ϕ might not be injective!

Example. If $E \in \mathrm{Coh}(C)$, there is a canonical object

$$\mathfrak{a}(E) = (E, H^0(C, E); \mathrm{id}).$$

The category $\mathcal{T}_C$

Set

$$\mathcal{T}_C = \{(E, V; \phi) : E \in \mathsf{Coh}(C),\ \phi\colon V \to H^0(C, E)\}.$$

These are *almost* coherent systems; note ϕ might not be injective!

Example. If $E \in \mathsf{Coh}(C)$, there is a canonical object

$$\mathfrak{a}(E) = (E, H^0(C, E); \mathrm{id}).$$

Note that $\mathcal{T}_C$ is an abelian category, so we can consider $\mathrm{D}^b(\mathcal{T}_C)$. In fact, we have $\mathrm{D}^b(\mathcal{T}_C) = \langle \mathrm{D}^b(pt), \mathrm{D}^b(C)\rangle$.

Exceptional objects

Alexeev and Kuznetsov: there are many *exceptional objects* on $\mathrm{D}^b(\mathcal{T}_C)$ arising from *BNP extremal line bundles*.

A line bundle L is **Brill–Noether–Petri extremal** if

$$H^0(C, L) \otimes H^0(C, L^{-1} \otimes \mathscr{O}_C(K_C)) \xrightarrow{\cong} H^0(C, \mathscr{O}_C(K_C))$$

L has *type* (r, s), $r = h^0(C, L)$ and $s = h^0(C, L^{-1} \otimes \mathscr{O}_C(K_C))$.

Exceptional objects

Alexeev and Kuznetsov: there are many *exceptional objects* on $\mathrm{D}^b(\mathcal{T}_C)$ arising from *BNP extremal line bundles*.

A line bundle L is **Brill–Noether–Petri extremal** if

$$H^0(C, L) \otimes H^0(C, L^{-1} \otimes \mathscr{O}_C(K_C)) \xrightarrow{\cong} H^0(C, \mathscr{O}_C(K_C))$$

L has *type* (r, s), $r = h^0(C, L)$ and $s = h^0(C, L^{-1} \otimes \mathscr{O}_C(K_C))$.

- $L = \mathscr{O}_C$ is BNP extremal, type $(1, g)$.
- $L = \omega_C$ is BNP extremal, type $(g, 1)$.
- If L is BNP extremal, then $rs = g$. If C is general, BNP extremal exist for each $rs = g$.
- (Alexeev–Kuznetsov) If L is BNP extremal, then $\mathfrak{a}(L) = (L, H^0(C, L); \mathrm{id})$ is exceptional.

FN stability conditions

Feyzbakhsh and Novik: $\mathrm{D}^b(\mathcal{T}_C)$ admits many interesting stability conditions.

Theorem (Feyzbakhsh–Novik, 2025)

There exists a two-dimensional family $\sigma_{b,w} = (Z_{b,w}, \mathcal{A}(b))$ of stability conditions on $\mathrm{D}^b(\mathcal{T}_C)$, with

$$Z_{b,w}(E) = -\mathbf{n}(E) + w\mathbf{r}(E) + i(\mathbf{d}(E) - b\mathbf{r}(E)).$$

These exists for $(b, w) \in \mathbb{R}^2$ satisfying $w > \Phi_C(b)$.

Note: If $F \in \mathrm{Coh}(C)$ is slope semistable, $\mu(F) = b$, then $\mathfrak{a}(F)[1] \in \mathcal{A}(b)$. Here

$$Z_{b,w}(\mathfrak{a}(F)[1]) = \mathbf{n}(F) - w\mathbf{r}(F),$$

so we need $w > \mathbf{n}(F)/\mathbf{r}(F)$.

By Bridgeland's deformation theorem, this needs to hold under small perturbations. We get

$$w > \lim_{\epsilon \to 0} \sup \left\{ \frac{h^0(C, F)}{\mathrm{rk}(F)} : F \text{ semistable, } \mu(E) \in (b - \epsilon, b + \epsilon) \right\}.$$

This is the mysterious function Φ_C.

By Bridgeland's deformation theorem, this needs to hold under small perturbations. We get

$$w > \lim_{\epsilon \to 0} \sup \left\{ \frac{h^0(C,F)}{\mathrm{rk}(F)} : F \text{ semistable, } \mu(E) \in (b-\epsilon, b+\epsilon) \right\}.$$

This is the mysterious function Φ_C.

Question. Compute Φ_C, or give good bounds.

Geometry of genus four curves

Now: C is a general genus four curve, $\phi_{|K_C|} : C \hookrightarrow \mathbb{P}^3$.

Facts:

- C is a $(2,3)$-complete intersection in $\mathbb{P}^3$.
- The quadric $Q \subset \mathbb{P}^3$ containing C is smooth; $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.
- $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(3,3)$.
- $L_1 = \mathscr{O}_Q(1,0)|_C$ and $L_2 = \mathscr{O}_Q(0,1)|_C$ are the only BNP extremal of type $(2,2)$.

(Higher rank) Brill–Noether theory

We have explicit bounds of Φ_C, by Lange and Newstead (2017). They show that:

$$\Phi_C(b) \leq \begin{cases} 1 & b = 0, \\ \frac{1}{4}b + \frac{3}{4} & 0 < b < 2, \\ \frac{1}{3}b + \frac{2}{3} & 2 \leq b < \frac{5}{2}, \\ \frac{1}{2}b + \frac{1}{4} & \frac{5}{2} \leq b < 3, \\ 2 & b = 3. \end{cases}$$

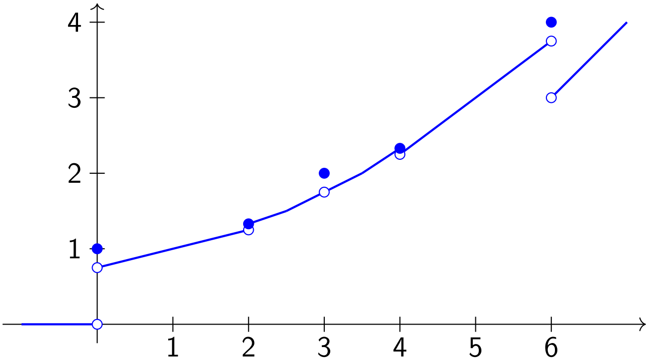

Figure: A bound of $w = \Phi_C(b)$.

Three “jumps”: $b = 0, 3, 6$. These are exactly the degrees of the BNP extremal line bundles. Moreover, only these line bundles attain the bound.

Question. Does Φ_C jump always at BNP extremal line bundles?

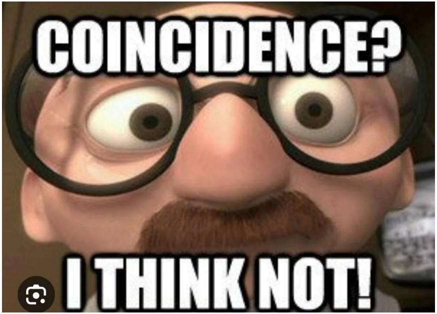

Figure: The speaker's reaction.

Question. Does Φ_C jump always at BNP extremal line bundles?

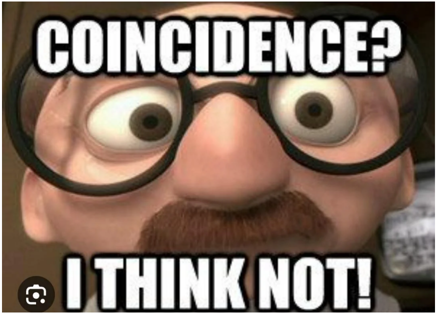

Figure: The speaker's reaction.

Question. What happens with $\mathrm{Stab}(\mathcal{T}_C)$ at the jump points?

Non-compactness

Key observation (Bridgeland): $\mathrm{Stab}(\mathscr{D})$ is non-compact!

- Obvious reason: rescale.
- More subtle: $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ is non-compact.

Non-compactness

Key observation (Bridgeland): $\mathrm{Stab}(\mathscr{D})$ is non-compact!

- Obvious reason: rescale.
- More subtle: $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ is non-compact.

Example. In $\mathrm{D}^b(C)$, take

$$Z_t(E) = -t\deg(E) + i\,\mathrm{rk}(E).$$

This does not converge in $\mathrm{Stab}(C)/\mathbb{C}$ as $t \to 0$.

General principle: $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ should admit a (partial?) compactification by “degenerate” stability conditions. Some keywords:

- *Weak* stability conditions: Toda, Collins–Lo–Shi–Yau, Broomhead–Pauksztello–Ploog–Woolf.
- *Polynomial* stability conditions: Bayer.
- *Augmented* stability conditions: Robotis–Halpern-Leistner.
- Bolognese's *metric compactification*.
- Bapat–Deopurkar–Licata's *Thurston compactification*.

Why do we care?

Two possible applications:

- Existence of stability conditions on Verdier quotients of $\mathscr{D}$.
- Moduli spaces and wall-crossing.

Back to C

Let us go back to our bound:

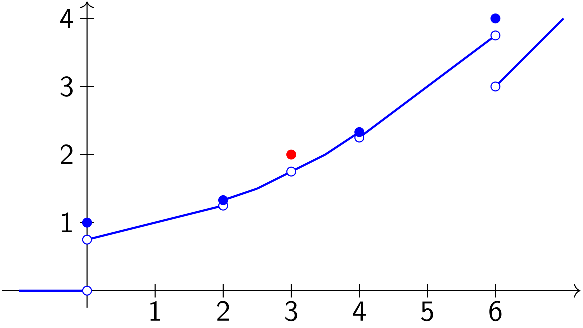

Figure: A bound of $w = \Phi_C(b)$.

By Feyzbakhsh–Novik, we have stability conditions $\sigma_{3,w}$ for $w > 2$.

Main result

Theorem (V., 2026)

The limit

$$\lim_{w\to 2^+} \sigma_{3,w} =: \overline{\sigma}_{3,2}$$

exists as a weak stability condition in $\mathrm{D}^b(\mathcal{T}_C)$. *Moreover,* $\ker Z_{3,2} = \langle \mathfrak{a}(L_1), \mathfrak{a}(L_2) \rangle$.

Here "weak" means "we allow objects in $\mathcal{A}$ to have $Z(E) = 0$."

Main result

Theorem (V., 2026)

The limit

$$\lim_{w \to 2^+} \sigma_{3,w} =: \overline{\sigma}_{3,2}$$

exists as a weak stability condition in $\mathrm{D}^b(\mathcal{T}_C)$. *Moreover,* $\ker Z_{3,2} = \langle \mathfrak{a}(L_1), \mathfrak{a}(L_2) \rangle$.

Here "weak" means "we allow objects in $\mathcal{A}$ to have $Z(E) = 0$."

As a consequence, we get a genuine stability condition on

$$\mathrm{D}^b(\mathcal{T}_C)/\langle \mathfrak{a}(L_1), \mathfrak{a}(L_2) \rangle.$$

What is this category?

Nodal cubic threefolds

Now: let $Y \subset \mathbb{P}^4$ be a nodal cubic threefold, $p \in Y$ node. Project:

$$\pi\colon Y \dashrightarrow \mathbb{P}^3.$$

Fact: π identifies $\mathrm{Bl}_p Y \cong \mathrm{Bl}_C \mathbb{P}^3$, where C is a canonically embedded genus 4 curve.

Nodal cubic threefolds

Now: let $Y \subset \mathbb{P}^4$ be a nodal cubic threefold, $p \in Y$ node. Project:

$$\pi\colon Y \dashrightarrow \mathbb{P}^3.$$

Fact: π identifies $\mathrm{Bl}_p Y \cong \mathrm{Bl}_C \mathbb{P}^3$, where C is a canonically embedded genus 4 curve.

On the other hand: semi-orthogonal decomposition

$$\mathrm{D}^b(Y) = \langle \mathrm{Ku}(Y), \mathscr{O}_Y(-1), \mathscr{O}_Y \rangle.$$

Theorem (Alexeev–Kuznetsov, 2025)

We have $\mathrm{Ku}(Y) \cong \mathrm{D}^b(\mathcal{T}_C)/\langle \mathfrak{a}(L_1), \mathfrak{a}(L_2) \rangle$.

Putting this together with our main result:

Corollary (V., 2026)

The weak stability condition $\overline{\sigma}_{3,2}$ *gives a stability condition* $\sigma \in \mathrm{Stab}(\mathrm{Ku}(Y))$.

Question. How this stability condition compares to other (potential) constructions? (Bernardara–Macrì–Mehrotra–Stellari and Bayer–Lahoz–Macrì–Stellari)

Future directions

Some questions to think about:

- Bound Φ_C for other curves, and describe where Φ_C jumps.
- Relate $\mathrm{D}^b(\mathcal{T}_C)$ for other curves with other Kuznetsov components of singular Fano varieties.
- Improve bounds of Φ_C for C genus four curve using nodal cubic threefolds, or vice versa?