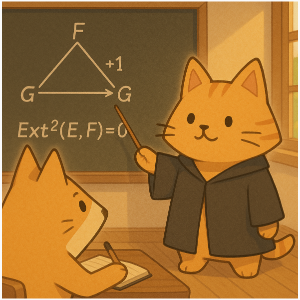

Derived Obsessed Graduate Students

Stability conditions on Kuznetsov components via quadric fibrations

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Introduction

Slope stability on curves	$\xrightarrow{\textit{generalise}}$	Bridegland stability on triangulated categroies
$E \in \mathrm{Coh}(C)$		$E \in \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}\, X)$

Introduction

$$\begin{array}{ccc} \text{Slope stability} & \xrightarrow{\textit{generalise}} & \text{Bridegland stability} \\ \text{on curves} & & \text{on triangulated categroies} \\ E \in \mathrm{Coh}(C) & & E \in \mathrm{D^b}(\mathrm{Coh}\, X) \end{array}$$

Stability conditions have been proved to exist on $\mathrm{D^b}$ of varieties...

- Varieties whose derived category admits a full exceptional collection: projective spaces $\mathbb{P}^n$, quadrics $Q^n \subseteq \mathbb{P}^{n+1}$, Grassmannians $\mathrm{Gr}(k, n)$.
- Curves (= slope stability);
- Surfaces (tilt stability);
- Fano threefolds;
- Abelian threefolds;
- Quintic threefolds, and some other complete intersection CY_3 in $\mathbb{P}^n$.

Kuznetsov components

Let $Y \subseteq \mathbb{P}^{n+1}$ be a smooth cubic n-fold over $\mathbb{C}$. The derived category $\mathrm{D}^{\mathrm{b}}(Y)$ admits the semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \dots, \mathcal{O}_Y(n-2) \rangle , \tag{1}$$

where the full subcategory

$$\mathcal{K}u(Y) = \left\{ E \in \mathrm{D}^{\mathrm{b}}(Y) \mid \forall\, i \in \{0, 1, \dots, n-2\},\ \mathrm{Ext}^{\bullet}(\mathcal{O}_Y(i), E) = 0 \right\}$$

is called the **Kuznetsov component** of Y.

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Known results

For a smooth cubic n-fold Y, there exists a Bridgeland stability condition on $\mathcal{K}u(Y)$:

- $n = 3$: Bernardara–Macrì–Mehrotra–Stellari, 2012;
- $n = 4$: Bayer–Lahoz–Macrì–Stellari, 2017;
- $n = 5$: *ongoing project*.

The current method cannot be generalised to $n \geqslant 6$!

Strategy of proof

- Find a linear subspace $\mathbb{P}^k \subseteq Y$ and construct a quadric fibration $\mathrm{Bl}_{\mathbb{P}^k} Y \longrightarrow \mathbb{P}^{n-k}$. →
- Construct a fully faithful functor $\mathcal{K}u(Y) \hookrightarrow \mathrm{D^b}(\mathbb{P}^{n-k}, \mathscr{C}_0)$ to a "twisted" derived category of $\mathbb{P}^{n-k}$. →
- Construct a weak stability condition on $\mathrm{D^b}(\mathbb{P}^{n-k}, \mathscr{C}_0)$. →
- Restrict the weak stability condition on $\mathcal{K}u(Y)$, which becomes a stability condition. →

Geometry of quadric fibrations

Geometric set-up

- Let V bs a $(n+2)$-dim vector space, $A \subseteq V$ a $(k+1)$-dim subspace, and $B := V/A$ the $(n-k+1)$-dim quotient space.

$$A \hookrightarrow V \twoheadrightarrow B$$

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- Blowing up $\mathbb{P}(V)$ along the k-plane $\Pi = \mathbb{P}(A) \subseteq Y$. Let E' be the exceptional divisor. Let $H' := \tau^*\mathscr{O}_{\mathbb{P}(V)}(1)$ and $h' := q^*\mathscr{O}_{\mathbb{P}(B)}(1)$.

$$
\begin{tikzcd}
 & E' \arrow[rr, hook] \arrow[dl] & & \widetilde{\mathbb{P}(V)} \arrow[dl, "\tau"'] \arrow[dr, "q"] & \\
\Pi \arrow[rr, hook] & & \mathbb{P}(V) \arrow[rr, dashed] & & \mathbb{P}(B)
\end{tikzcd}
$$

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- $q : \operatorname{Bl}_\Pi \mathbb{P}(V) \to \mathbb{P}(B)$ is a $\mathbb{P}^{k+1}$-fibration. $\operatorname{Bl}_\Pi \mathbb{P}(V) = \mathbb{P}_{\mathbb{P}(B)}(\mathcal{F})$, where

$$\mathcal{F} := \left(q_*\tau^*\mathcal{O}_{\mathbb{P}(V)}(1)\right)^{\vee} \cong \mathcal{O}_{\mathbb{P}(B)}^{\oplus(k+1)} \oplus \mathcal{O}_{\mathbb{P}(B)}(-1)$$

  is locally free of rank $(k+2)$.

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- Let $Y \subseteq \mathbb{P}(V)$ is a smooth cubic n-fold with $\Pi \subseteq Y$. Consider the embedded blow-up. Note that $H' - E' = h'$ and $3H' - E' = \widetilde{Y}$ in $\operatorname{Pic} \mathbb{P}(\mathcal{F})$.

$$\begin{tikzcd}
 & E \arrow[r, hook, "\iota"] \arrow[dl, "p"'] & \widetilde{Y} \arrow[r, hook, "\alpha"] \arrow[dl, "\sigma"'] \arrow[drr, "\pi"] & \mathbb{P}(\mathcal{F}) \arrow[dl, "\tau"'] \arrow[dr, "q"] & \\
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- $q_* \mathcal{O}_{\mathbb{P}(\mathcal{F})}(\widetilde{Y}) = q_* \mathcal{O}_{\mathbb{P}(\mathcal{F})}(2H' + h') \cong \operatorname{Sym}^2 \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}(B)}(1)$.
  Hence $\widetilde{Y}$ is defined by a section of $\operatorname{Sym}^2 \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}(B)}(1)$, or a quadratic form $Q : \mathcal{F} \longrightarrow \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}(B)}(1)$.

  $\pi = q \circ \alpha : \widetilde{Y} \longrightarrow \mathbb{P}(B)$ is a fibration in k-dimensional quadrics. Back

$$\begin{tikzcd}
 & E \arrow[rr, hook, "\iota"] \arrow[dl, "p"'] & & \widetilde{Y} \arrow[rr, hook, "\alpha"] \arrow[dl, "\sigma"'] \arrow[drrr, "\pi"] & & \mathbb{P}(\mathcal{F}) \arrow[dl, "\tau"'] \arrow[dr, "q"] & \\
\Pi \arrow[rr, hook] & & Y \arrow[rr, hook] & & \mathbb{P}(V) \arrow[rr, dashed] & & \mathbb{P}(B)
\end{tikzcd}$$

Clifford sheaves (*Kuznetsov, 2008*)

Quadratic form $Q \rightsquigarrow$ sheaf of Clifford algebras on $\mathbb{P}(B)$.

Even & odd parts:

$$\mathscr{C}_0 := \bigoplus_{m=0}^{\infty} \bigwedge^{2m} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(B)}(m), \qquad \mathscr{C}_1 := \bigoplus_{m=0}^{\infty} \bigwedge^{2m+1} \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(B)}(m).$$

Set $\mathscr{C}_{2j} := \mathscr{C}_0 \otimes \mathcal{O}_{\mathbb{P}(B)}(j)$ and $\mathscr{C}_{2j+1} := \mathscr{C}_1 \otimes \mathcal{O}_{\mathbb{P}(B)}(j)$ for any $j \in \mathbb{Z}$.

$\mathscr{C}_j$ are flat right $\mathscr{C}_0$-modules. *Think of them as "line bundles" on $(\mathbb{P}(B), \mathscr{C}_0)$, as $\mathscr{C}_j \otimes_{\mathscr{C}_0} \mathscr{C}_k \cong \mathscr{C}_{j+k}$.*

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Denote by

- $\mathrm{Coh}(\mathbb{P}(B), \mathscr{C}_0)$ the Abelian category of coherent right $\mathscr{C}_0$-modules;
- $\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B), \mathscr{C}_0)$ the derived category of coherent right $\mathscr{C}_0$-modules.

The forgetful functor $\mathrm{Forg}: \mathrm{D}^{\mathrm{b}}(\mathbb{P}(B), \mathscr{C}_0) \to \mathrm{D}^{\mathrm{b}}(\mathbb{P}(B))$ admits both left and right adjoints:

$$(- \otimes_{\mathscr{O}_{\mathbb{P}(B)}} \mathscr{C}_0) \dashv \mathrm{Forg} \dashv (- \otimes_{\mathscr{O}_{\mathbb{P}(B)}} \mathscr{C}_0^{\vee}). \tag{2}$$

The Serre functor of the category $\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B), \mathscr{C}_0)$:

$$\mathrm{S}(E) = \omega_{\mathbb{P}(B)} \otimes_{\mathscr{O}} E \otimes_{\mathscr{C}_0} \mathscr{C}_0^{\vee}[n-k].$$

Semi-orthogonal decompositions and mutations of $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$

Semi-orthogonal decompositions on $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$

$$\begin{tikzcd}
& E \arrow[r, hook, "\iota"] \arrow[dl, "p"'] & \widetilde{Y} \arrow[r, hook, "\alpha"] \arrow[dl, "\sigma"'] \arrow[drr, "\pi"] & \mathbb{P}(\mathcal{F}) \arrow[dl, "\tau"', crossing over] \arrow[dr, "q"] & \\
\Pi \arrow[r, hook] & Y \arrow[r, hook] & \mathbb{P}(V) \arrow[rr, dashed] & & \mathbb{P}(B)
\end{tikzcd}$$

- The derived pull-back $\sigma^* : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(\widetilde{Y})$ is fully faithful. Orlov's formula gives the SOD of $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$:

$$\begin{aligned}
&\Big\langle \sigma^*\mathrm{D}^{\mathrm{b}}(Y),\ \iota_* p^*\mathrm{D}^{\mathrm{b}}(\Pi),\ \iota_*(p^*\mathrm{D}^{\mathrm{b}}(\Pi)\otimes\mathcal{O}_E(-E)),\ \dots,\ \iota_*(p^*\mathrm{D}^{\mathrm{b}}(\Pi)\otimes\mathcal{O}_E(-(n-k-2)E))\Big\rangle \\
&= \langle \sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \dots,\ \mathcal{O}((n-2)H),\ \iota_*\mathcal{O}_E,\ \dots,\ \iota_*\mathcal{O}_E(kH), \dots, \\
&\qquad\qquad \iota_*\mathcal{O}_E(-(n-k-2)E),\ \dots,\ \iota_*\mathcal{O}_E(kH-(n-k-2)E)\rangle.
\end{aligned}$$

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&= \langle \sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \dots,\ \mathcal{O}((n-2)H),\ \iota_*\mathcal{O}_E,\ \dots,\ \iota_*\mathcal{O}_E(kH), \dots, \\
&\qquad\qquad \iota_*\mathcal{O}_E(-(n-k-2)E),\ \dots,\ \iota_*\mathcal{O}_E(kH-(n-k-2)E)\rangle.
\end{aligned}$$

- Fully faithful functor $\Phi : \mathrm{D}^{\mathrm{b}}(\mathbb{P}(B),\mathscr{C}_0) \to \mathrm{D}^{\mathrm{b}}(\widetilde{Y})$, $\Phi(F) = \pi^*F \otimes_{\pi^*\mathscr{C}_0} \mathcal{E}'$, where $\mathcal{E}'$ is a $\pi^*\mathscr{C}_0$-module that fits into the SES:

$$0 \longrightarrow q^*\mathscr{C}_0(-2H) \longrightarrow q^*\mathscr{C}_1(-H) \longrightarrow \alpha_*\mathcal{E}' \longrightarrow 0$$

  There is an SOD of $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$ [*Kuznetsov, 2008*]:

$$\begin{aligned}
&\left\langle \Phi\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B),\mathscr{C}_0),\ \pi^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B)),\ \pi^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B))\otimes\mathcal{O}(H),\ \dots,\ \pi^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B))\otimes\mathcal{O}((k-1)H)\right\rangle \\
&= \langle \Phi\mathrm{D}^{\mathrm{b}}(\mathbb{P}(B),\mathscr{C}_0),\ \mathcal{O},\ \dots,\ \mathcal{O}((n-k)h),\ \dots, \mathcal{O}((k-1)H),\ \dots,\ \mathcal{O}((k-1)H+(n-k)h)\rangle.
\end{aligned}$$

Mutations on $\mathrm{D^b}(\widetilde{Y})$

Comparing the two SODs, for small (n, k) it is possible to use a sequence of mutations to transform one into another:

- $(n, k) = (3, 1)$: the blow-up along a line in a cubic 3-fold induces a conic fibration. $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0) \simeq \left\langle Ku(Y^3),\ \mathcal{B}_1 \right\rangle$. [*BMMS, 2012*].

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- $(n,k) = (4,1)$: the blow-up along a line in a cubic 4-fold induces a conic fibration. $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) \simeq \left\langle Ku(Y^4),\ \mathcal{B}_1,\ \mathcal{B}_2,\ \mathcal{B}_3 \right\rangle$. [*BLMS, 2017*].

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- $(n,k) = (4,2)$: the blow-up along the plane in a special cubic 4-fold containing that plane:

$$\mathrm{D^b}(S,\alpha) \simeq \mathrm{D^b}(\mathbb{P}^2, \mathscr{C}_0) \simeq \mathcal{K}u(Y^4). \qquad [\textit{Kuznetsov, 2009}]$$

$\mathcal{K}u(Y^4)$ is considered as a non-commutative K3 surface. Kuznetsov's rationality conjecture: a smooth cubic 4-fold Y^4 is rational iff $\mathcal{K}u(Y^4) \simeq \mathrm{D^b}(S)$ for some K3 surface S.

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- $(n,k)=(5,2)$: the blow-up along a plane in a cubic 5-fold induces a quadric surface fibration. $\mathrm{D}^b(\mathbb{P}^3,\mathscr{C}_0)\simeq\left\langle\mathcal{K}u(Y^5),\ \mathscr{C}_1,\ \mathscr{C}_2\right\rangle$. (*new result*)

Back

Mutations on $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$

Mutations on a cubic 5-fold takes 14 steps!

$$
\begin{aligned}
&\mathrm{D}^{\mathrm{b}}(\widetilde{Y})\\
&=\langle \mathcal{O}(-H),\ \sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(H),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E,\ \iota_*\mathcal{O}_E(H),\ \iota_*\mathcal{O}_E(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E)\rangle\\
&=\langle \mathcal{O}(-H),\ \sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \iota_*\mathcal{O}_E,\ \mathcal{O}(H),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(H),\ \iota_*\mathcal{O}_E(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E)\rangle\\
&\qquad\text{(Left mutation of } \iota_*\mathcal{O}_E \text{ through } \langle \mathcal{O}(H),\mathcal{O}(2H)\rangle\text{, using Lemma 2.5.(i))}\\
&=\langle \mathcal{O}(-H),\ \sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \iota_*\mathcal{O}_E,\ \mathcal{O}(H),\ \iota_*\mathcal{O}_E(H),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E)\rangle\\
&\qquad\text{(Left mutation of } \iota_*\mathcal{O}_E(H) \text{ through } \mathcal{O}(2H)\text{, using Lemma 2.5.(i))}\\
&=\langle \mathcal{O}(-H),\ \sigma^*\mathcal{K}u(Y),\ \mathcal{O}(-H+h),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E)\rangle\\
&\qquad\text{(Left mutation of } \iota_*\mathcal{O}_E(aH) \text{ through } \mathcal{O}(aH) \text{ for } a=0,1,2\text{, using Lemma 2.5.(ii))}\\
&=\left\langle \mathcal{O}(-H),\ \mathcal{O}(-H+h),\ \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E)\right\rangle\\
&=\left\langle \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(H-E),\ \iota_*\mathcal{O}_E(2H-E),\ \iota_*\mathcal{O}_E(3H-E),\ \mathcal{O}(H+2h),\ \mathcal{O}(H+3h)\right\rangle\\
&\qquad\text{(Right mutation of } \langle \mathcal{O}(-H),\mathcal{O}(-H+h)\rangle \text{ through its left orthogonal, using the Serre functor } \mathrm{S}=(-\otimes\mathcal{O}(-2H-2h))[5])\\
&=\left\langle \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(h),\ \iota_*\mathcal{O}_E(H+h),\ \iota_*\mathcal{O}_E(2H+h),\ \mathcal{O}(H+2h),\ \mathcal{O}(H+3h)\right\rangle\\
&\qquad(E=H-h)\\
&=\left\langle \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(2H),\ \iota_*\mathcal{O}_E(h),\ \iota_*\mathcal{O}_E(H+h),\ \mathcal{O}(H+2h),\ \mathcal{O}(2H+h),\ \mathcal{O}(H+3h)\right\rangle\\
&\qquad\text{(Right mutation of } \iota_*\mathcal{O}_E(2H+h) \text{ through } \mathcal{O}(H+2h)\text{, using Lemma 2.5.(iii))}\\
&=\left\langle \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(H+h),\ \mathcal{O}(2H),\ \mathcal{O}(H+2h),\ \mathcal{O}(2H+h),\ \mathcal{O}(H+3h)\right\rangle\\
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&\qquad\text{(Left mutation of } \mathcal{O}(H+2h) \text{ through } \mathcal{O}(2H) \text{ and } \mathcal{O}(H+3h) \text{ through } \mathcal{O}(2H+h)\text{, using Lemma 2.5.(iv))}\\
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&\qquad\text{(Left mutation of } \mathcal{O}(H+3h) \text{ through } \mathcal{O}(2H)\text{, using Lemma 2.5.(iv))}\\
&=\left\langle \mathcal{O}(-2h),\ \mathcal{O}(-h),\ \mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(H+h),\ \mathcal{O}(H+2h),\ \mathcal{O}(H+3h)\right\rangle\\
&\qquad\text{(Left mutation of } \langle \mathcal{O}(2H),\mathcal{O}(2H+h)\rangle \text{ through its right orthogonal, using the Serre functor)}\\
&=\left\langle \mathbf{L}_{\mathcal{O}(-2h)}\mathbf{L}_{\mathcal{O}(-h)}\mathbf{R}_{\mathcal{O}(-H+h)}\sigma^*\mathcal{K}u(Y),\ \mathcal{O}(-2h),\ \mathcal{O}(-h),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(H+h),\ \mathcal{O}(H+2h),\ \mathcal{O}(H+3h)\right\rangle\\
&=\langle \mathcal{K},\ \mathcal{O}(-2h),\ \mathcal{O}(-h),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h),\ \mathcal{O}(H+2h),\ \mathcal{O}(H+3h)\rangle\\
&\qquad\text{(Left mutation of } \left\langle \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(h),\ \mathbf{L}_{\mathcal{O}(2H)}\iota_*\mathcal{O}_E(H+h)\right\rangle \text{ through } \langle \mathcal{O}(-2h),\ \mathcal{O}(-h),\ \mathcal{O},\ \mathcal{O}(h),\ \mathcal{O}(H),\ \mathcal{O}(H+h)\rangle)
\end{aligned}
$$

Mutations on $\mathrm{D^b}(\widetilde{Y})$

Mutations on a cubic 5-fold takes 14 steps!

Not knowing of a general method, we conjecture that the fibration $\mathrm{Bl}_{\mathbb{P}^k} Y^n \to \mathbb{P}^{n-k}$ in k-dimensional quadrics induces the equivalence

$$\mathrm{D^b}(\mathbb{P}^{n-k}, \mathscr{C}_0) \simeq \langle \mathcal{K}u(Y^n),\ \mathscr{C}_1,\ \ldots,\ \mathscr{C}_{2n-3k-2} \rangle$$

for all possible (n, k) with $k \leqslant n/2$.

Bridgeland stability conditions

Categorical set-up

- $\mathcal{D}$ a $\mathbb{C}$-linear triangulated category, and $\mathrm{K}_0(\mathcal{D})$ its Grothendieck group.
- Fix a finite-rank lattice Λ and a surjective group homomorphism $v : \mathrm{K}_0(\mathcal{D}) \to \Lambda$.
- $\mathcal{A}$ the heart of a bounded t-structure on $\mathcal{D}$.

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Definition

A **weak stability function** is a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ such that, for any $E \in \mathcal{A} \setminus \{0\}$,

$$Z(v(E)) \in \left\{z = m \cdot \mathrm{e}^{\mathrm{i}\pi\phi} \mid m \geqslant 0,\ \phi \in (0,1]\right\} = \mathbb{H} \cup \mathbb{R}_{\leqslant 0}.$$

$\phi = \phi(E)$ is called the phase of E. If we require further that $Z(v(E)) \neq 0$ for $E \neq 0$, then Z is called a **stability function**.

An object $E \in \mathcal{A}$ is called **semi-stable** (*resp.* **stable**) with respect to $(\mathcal{A}, Z)$, if for any $F \hookrightarrow E$ with $F \ncong E$ in $\mathcal{A}$, one has $\phi(F) \leqslant \phi(E/F)$ (*resp.* $<$).

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The slope of E with respect to Z:

$$\mu_Z(E) = -\cot(\pi\phi(E)) = \begin{cases} -\frac{\operatorname{Re} Z(v(E))}{\operatorname{Im} Z(v(E))}, & \operatorname{Im} Z(v(E)) > 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

(Weak) stability conditions

Definition

A **(weak) stability condition** on $\mathcal{D}$ wth respect to Λ is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z_{\mathcal{A}} : \Lambda \to \mathbb{C}$ is a (weak) stability function satisfying:

(i) (**Harder–Narasimhan property**) For any $E \in \mathcal{A}$, there exists a filtration

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell =: E$$

by objects E_i in $\mathcal{A}$, such that the graded factors E_i/E_{i-1} are semi-stable of phase ϕ_i, and

$$\phi^+(E) := \phi_1 > \cdots > \phi_\ell =: \phi^-(E).$$

(ii) (**Support property**) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(E) \geqslant 0$ for semi-stable $E \in \mathcal{A}$.

The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold of dimension equal to $\operatorname{rk} \Lambda$, a celebrated result by Bridgeland.

Slope stability

Let X be an n-dimensional smooth projective variety, and $h \in \operatorname{Pic} X$ an ample divisor class.

Example

The classical slope stability on $\mathrm{D}^{\mathrm{b}}(X)$ is $(\mathrm{Coh}(X), Z_\mu)$, where

$$Z_\mu(E) = -h^{n-1} \cdot \mathrm{ch}_1(E) + \mathrm{i}h^n \cdot \mathrm{ch}_0(E),$$

is a weak stability condition with respect to the lattice generated by $(\mathrm{ch}_0, \mathrm{ch}_1)$.

- The slope $\mu(E) = -\frac{\operatorname{Re} Z(v(E))}{\operatorname{Im} Z(v(E))} = \frac{\deg E}{h^n \operatorname{rk} E}$ is directly proportional to the classical slope.
- The support property is satisfied by the trivial quadratic form $Q = 0$.

Tilting stability conditions

Given the heart of a bounded t-structure $\mathcal{A}$ of $\mathcal{D}$ and $\mu \in \mathbb{R}$, consider the full subcategories

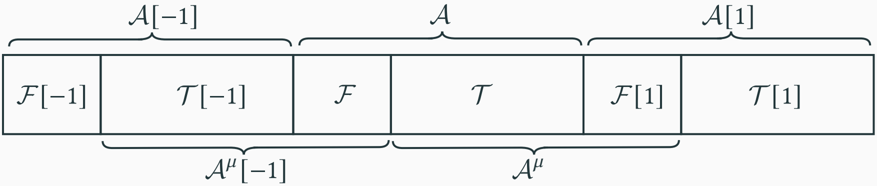
$$\mathcal{T}^{\mu} = \left\{E \in \mathcal{A} \mid \mu_Z^{-}(E) > \mu\right\}; \qquad \mathcal{F}^{\mu} \left\{E \in \mathcal{A} \mid \mu_Z^{+}(E) \leqslant \mu\right\}.$$

Then $(\mathcal{T}^{\mu}, \mathcal{F}^{\mu})$ is a torsion pair. The tilt

$$\mathcal{A}^{\mu} := \langle \mathcal{T}^{\mu},\ \mathcal{F}^{\mu}[1] \rangle$$

is a heart of bounded t-structure of $\mathcal{D}$. The objects $E \in \mathcal{A}^{\mu}$ has cohomology

$$\mathcal{H}^{i}(E) \begin{cases} \in \mathcal{T}, & i = 0; \\ \in \mathcal{F}, & i = -1; \\ = 0, & \text{otherwise.} \end{cases}$$



Tilting stability conditions

Let X be an n-dimensional smooth projective variety, and $h \in \operatorname{Pic} X$ an ample divisor class.

Example

For $\beta \in \mathbb{R}$, we can tilt the heart $\operatorname{Coh}(X)$ at $\mu = \beta$ to get a new heart $\operatorname{Coh}^{\beta}(X)$. Then the (first) tilt stability $(\operatorname{Coh}^{\beta}(X), Z_{\alpha,\beta})$ where

$$Z_{\alpha,\beta}(E) := -h^{n-2} \cdot \left(\operatorname{ch}_2^{\beta}(E) - \frac{1}{2}\alpha^2 \operatorname{ch}_0^{\beta}(E)\right) + \mathrm{i} h^{n-1} \cdot \operatorname{ch}_1^{\beta}(E),$$

where $\operatorname{ch}^{\beta}(E) := \mathrm{e}^{-\beta h} \cdot \operatorname{ch}(E)$, is a weak stability condition with respect to the lattice generated by $(\operatorname{ch}_0, \operatorname{ch}_1, \operatorname{ch}_2)$.

The support property is given by the **Bogomolov's inequality**:

$$\Delta(E) := h^{n-2} \cdot (\operatorname{ch}_1(E)^2 - 2\operatorname{ch}_0(E)\operatorname{ch}_2(E)) \geqslant 0,$$

which holds for μ-semistable torsion-free sheaves E.

Stability conditions on $\mathcal{K}u(Y)$

Weak tilt stability on $\mathrm{D}^b(\mathrm{P}^3, \mathscr{C}_0)$

Fix Y smooth cubic 5-fold, $\Pi \subseteq Y$ a 2-plane. The quadric surface fibration $\mathrm{Bl}_\Pi Y \to \mathbb{P}^3$ induces:

$$\mathrm{D}^b(\mathbb{P}^3, \mathscr{C}_0) \simeq \langle \mathcal{K}u(Y), \mathscr{C}_1, \mathscr{C}_2 \rangle \,; \quad \mathscr{C}_0 = \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 3} \oplus \mathcal{O}(-2)^{\oplus 3} \oplus \mathcal{O}(-3).$$

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- On $\mathrm{D}^b(\mathbb{P}^3, \mathscr{C}_0)$ we have the slope stability:

$$(\mathrm{Coh}(\mathbb{P}^3, \mathscr{C}_0),\ Z = -h^2 \operatorname{ch}_1 + \mathrm{i} h^3 \operatorname{ch}_0).$$

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- Tilt it at $\mu = \beta$: consider the pair $\sigma_{\alpha,\beta} = (\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathscr{C}_0), Z_{\alpha,\beta})$ with

$$Z_{\alpha,\beta}(E) := -h\left(\operatorname{ch}^{\beta}_{\mathscr{C}_0,2}(E) - \frac{1}{2}\alpha^2 \operatorname{ch}^{\beta}_{\mathscr{C}_0,0}(E)\right) + \mathrm{i}h^2 \operatorname{ch}^{\beta}_{\mathscr{C}_0,1}(E).$$

Here $\operatorname{ch}^{\beta}_{\mathscr{C}_0}$ is a **modified Chern character**, defined by

$$\operatorname{ch}^{\beta}_{\mathscr{C}_0}(E) := \mathrm{e}^{-\beta h}\left(1 - \frac{3}{8}h^2\right) \operatorname{ch}(\mathrm{Forg}(E)).$$

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- For support property, we need the modified Bogomolov inequality

$$\begin{aligned} \Delta_{\mathscr{C}_0}(E) &:= \operatorname{ch}_{\mathscr{C}_0,1}(E)^2 - 2 \operatorname{ch}_{\mathscr{C}_0,0}(E) \operatorname{ch}_{\mathscr{C}_0,2}(E) \\ &= \operatorname{ch}_1(E)^2 - 2 \operatorname{ch}_0(E) \operatorname{ch}_2(E) + \frac{3}{4} h^2 \operatorname{ch}_0(E) \geqslant 0. \end{aligned}$$

Back

Weak tilt stability on $\mathrm{D}^{\mathrm{b}}(\mathrm{P}^3, \mathscr{C}_0)$

$$Z_{\alpha,\beta}(E) := -h^{n-2} \cdot \left(\mathrm{ch}_2^{\beta}(E) - \frac{1}{2}\alpha^2 \mathrm{ch}_0^{\beta}(E)\right) + \mathrm{i}h^{n-1} \cdot \mathrm{ch}_1^{\beta}(E),$$

$$\mathrm{ch}_{\mathscr{C}_0}(E) := \left(1 - \frac{3}{8}h^2\right) \mathrm{ch}(\mathrm{Forg}(E)).$$

The number $-\frac{3}{8}$ is chosen such that $\Delta_{\mathscr{C}_0}(\mathscr{C}_j) = 0$ for all $j \in \mathbb{Z}$.

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To prove $\Delta_{\mathscr{C}_0}(E) \geqslant 0$ for μ-semistable E:

- [*BLMS*] Use a Langer-type restriction theorem to restrict on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathscr{C}_0)$ (*the difficult part*);

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- Inequality follows from Hirzebruch–Riemann–Roch on $\mathbb{P}^2$ and the fact that every $\mathscr{C}_0$-module has rank divisible by 8.

It is still a *weak* stability condition as all 0-dim sheaves lie in $\ker Z_{\alpha,\beta}$.

Inducing stability conditions on $\mathcal{K}u(Y)$

Proposition ([*BLMS, Proposition 5.1*])

Let $\sigma = (\mathcal{A}, Z)$ be a weak stability condition on $\mathcal{D}$ with a Serre functor S. Assume that $\mathcal{D} = \langle \mathcal{D}_1, E_1, \dots, E_m \rangle$, where $E_i \in \mathcal{D}$ are exceptional objects. Then $\sigma_1 = (\mathcal{A} \cap \mathcal{D}_1, Z|_{\mathcal{D}_1})$ is a stability condition on $\mathcal{D}_1$, if for $i = 1, \dots, m$ the following conditions are satisfied:

(1) $E_i \in \mathcal{A}$;
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- For suitable (α, β), we have

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- Tilt $\mathrm{Coh}^{\beta}(\mathbb{P}^3, \mathscr{C}_0)$ at $\nu_{\alpha,\beta} = 0$: $\mathrm{Coh}^0_{\alpha,\beta}(\mathbb{P}^3, \mathscr{C}_0) = \left\langle \mathcal{T}^0_{\alpha,\beta},\ \mathcal{F}^0_{\alpha,\beta}[1] \right\rangle$.

  $\sigma^0_{\alpha,\beta} = (\mathrm{Coh}^0_{\alpha,\beta}(\mathbb{P}^3, \mathscr{C}_0),\ Z^0_{\alpha,\beta} = -\mathrm{i}Z_{\alpha,\beta})$ is a *rotation of* $\sigma_{\alpha,\beta}$ *by* $3\pi/2$.

Stability conditions on $\mathcal{K}u(Y)$

Theorem

Let Y be a smooth cubic 5-fold. $\mathcal{K}u(Y)$ has a family of Bridgeland stability conditions

$$\sigma'_{\alpha,\beta} = \left(\mathrm{Coh}^0_{\alpha,\beta}(\mathbb{P}^3, \mathscr{C}_0) \cap \mathcal{K}u(Y),\ Z^0_{\alpha,\beta}|_{\mathcal{K}u(Y)}\right),$$

parametrised by $\left\{(\alpha,\beta) \in \mathbb{R}^2 \mid -\frac{3}{2} < \beta < -1,\ 0 < \alpha < \min\left\{\beta + \frac{3}{2}, -1-\beta\right\}\right\}$.

It is a stability condition as $\ker Z^0_{\alpha,\beta}$ *is generated by* $\mathrm{Coh}_{\dim=0}(\mathbb{P}^3, \mathscr{C}_0)$ *and* $\mathrm{Coh}_{\dim=0}(\mathbb{P}^3, \mathscr{C}_0) \cap \mathcal{K}u(Y) = \{0\}$.

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Let Y be a smooth cubic 5-fold. $\mathcal{K}u(Y)$ has a family of Bridgeland stability conditions

$$\sigma'_{\alpha,\beta} = \left(\mathrm{Coh}^0_{\alpha,\beta}(\mathbb{P}^3, \mathscr{C}_0) \cap \mathcal{K}u(Y),\ Z^0_{\alpha,\beta}|_{\mathcal{K}u(Y)}\right),$$

parametrised by $\left\{(\alpha,\beta) \in \mathbb{R}^2 \mid -\frac{3}{2} < \beta < -1,\ 0 < \alpha < \min\left\{\beta + \frac{3}{2}, -1-\beta\right\}\right\}$.

It is a stability condition as $\ker Z^0_{\alpha,\beta}$ *is generated by* $\mathrm{Coh}_{\dim=0}(\mathbb{P}^3, \mathscr{C}_0)$ *and* $\mathrm{Coh}_{\dim=0}(\mathbb{P}^3, \mathscr{C}_0) \cap \mathcal{K}u(Y) = \{0\}$.

Further remarks:

- By Collins–Polishchuk gluing, $\mathrm{D}^b(Y) = \langle \mathcal{K}u(Y), \mathscr{O}_Y, \ldots, \mathscr{O}_Y(3)\rangle$ has non-empty $\mathrm{Stab}(\mathrm{D}^b(\mathrm{Y}))$.
- The $\sigma'_{\alpha,\beta}$-(semi)stability of any $E \in \mathcal{K}u(Y)$ is independent of α, β, and the choice of the 2-plane $\Pi \subseteq Y$.

Back

Classical and Bridgeland moduli spaces

The lattice of $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(Y))$

For a cubic n-fold Y^n, the Serre functor S of $\mathcal{K}u(Y^n)$ satisfies $\mathrm{S}^3 \simeq [n+2]$.

- $\mathcal{K}u(Y^3)$ is a fractional $\frac{5}{3}$-CY category.
- $\mathcal{K}u(Y^4)$ is a CY_2 category (= K3 category).
- $\mathcal{K}u(Y^5)$ is a fractional $\frac{7}{3}$-CY category.

The stability condition σ' on $\mathcal{K}u(Y^n)$ is **Serre invariant**: $\mathrm{S} \cdot \sigma' \subseteq \sigma' \cdot \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

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For smooth cubic 3-folds, 5-folds and general cubic 4-folds, $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(Y))$ is a rank 2 lattice spanned by the characters

$$\kappa_1 = [\mathrm{pr}(\mathscr{I}_\Pi)], \quad \kappa_2 = -[\mathrm{pr}(\mathscr{I}_\Pi(1))].$$

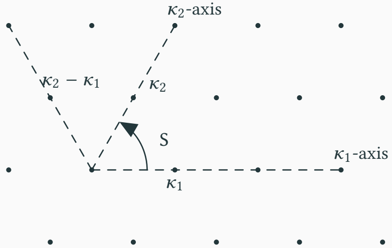

Figure 1: Characters in $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(Y^5))$ under the hexagonal coordinate.

Moduli spaces on $\mathcal{K}u(Y)$

There are some well-known results on cubic 3-folds.

Theorem (*BMMS 2009, Pertusi–Yang 2020, Feyzbakhsh–Pertusi 2023*)

Let Y be a smooth cubic 3-fold. Then there are isomorphisms of moduli spaces

$$M_{\sigma'}(\mathcal{K}u(Y), \kappa_1) \cong M_{\sigma'}(\mathcal{K}u(Y), \kappa_2) \cong M_{\sigma'}(\mathcal{K}u(Y), \kappa_2 - \kappa_1) \cong \mathcal{F}_1(Y),$$

where $M_{\sigma'}(\mathcal{K}u(Y), \kappa_1)$ is the moduli space of σ'-stable objects in $\mathcal{K}u(Y)$ with character κ_1, and $\mathcal{F}_1(Y)$ is the Fano surface of lines in Y.

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Let Y, Y' be two cubic 3-folds. Then $Y \cong Y' \iff \mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$

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$$\text{Idea:}\quad \begin{aligned} \mathcal{K}u(Y) \simeq \mathcal{K}u(Y') &\implies M_\sigma(\mathcal{K}u(Y), [\mathcal{I}_\ell]) \cong M_{\sigma'}(\mathcal{K}u(Y'), [\mathcal{I}_{\ell'}]) \\ &\implies \mathcal{F}_1(Y) \cong \mathcal{F}_1(Y') \\ &\implies Y \cong Y' \quad \text{(geometric Torelli).} \end{aligned}$$

Moduli spaces on $\mathcal{K}u(Y)$

Cubic 4-folds are more interesting from their connection with K3 surfaces and hyper-Kähler manifolds.

Theorem (*Bayer–Lahoz–Macrì–Nuer–Perry–Stellari, 2021*)

Let Y be a cubic 4-fold. For a character v in the Mukai–Hodge lattice $\widetilde{\mathrm{H}}(\mathcal{K}u(Y), \mathbb{Z})$ and a stability condition $\sigma \in \mathrm{Stab}^{\dagger}(\mathcal{K}u(Y))$, the moduli space $M_{\sigma}(\mathcal{K}u(Y), v)$ is a smooth projective hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type with dimension equal to $2 - \chi(v, v)$.

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Theorem (Categorical Torelli theorem for cubic 4-folds) (*BLMS 2017, Li–Pertusi–Zhao 2020*)

Let Y, Y' be two cubic 4-folds. Then $Y \cong Y'$ iff there is an equivalence $\mathcal{K}u(Y) \to \mathcal{K}u(Y')$ whose induced map $\widetilde{\mathrm{H}}(\mathcal{K}u(Y), \mathbb{Z}) \to \widetilde{\mathrm{H}}(\mathcal{K}u(Y'), \mathbb{Z})$ commutes with the degree shift functor $\mathrm{L}_{\mathcal{O}}(- \otimes \mathcal{O}(1))$.

Thank you for your attention!

***D**erived **O**bsessed **G**raduate **S**tudents*