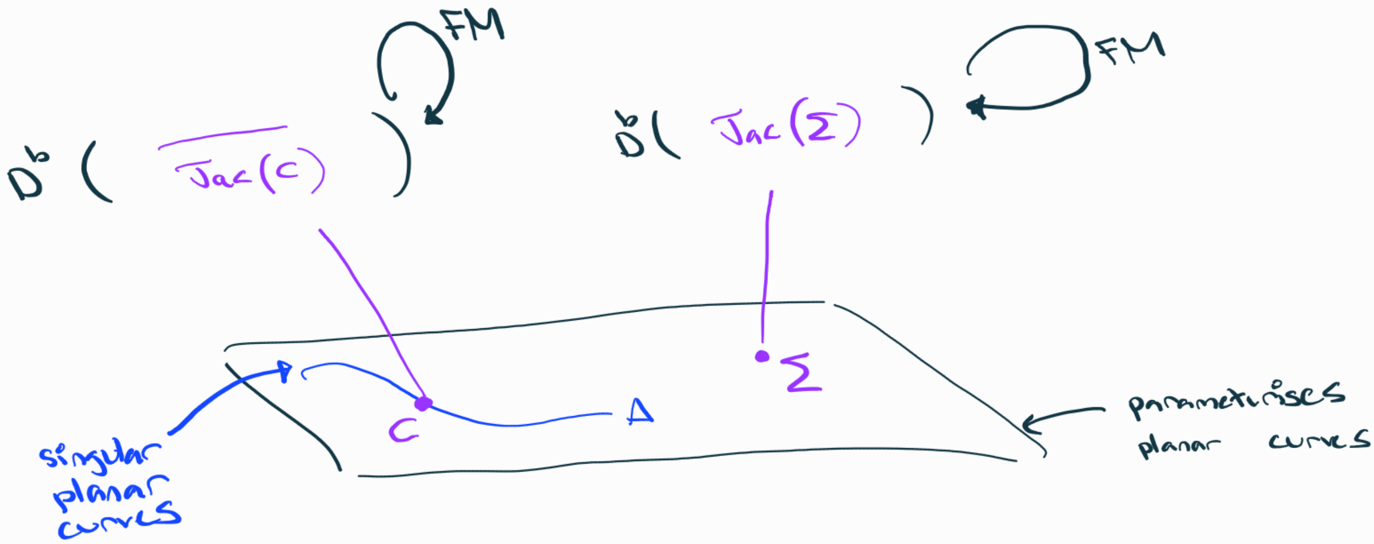
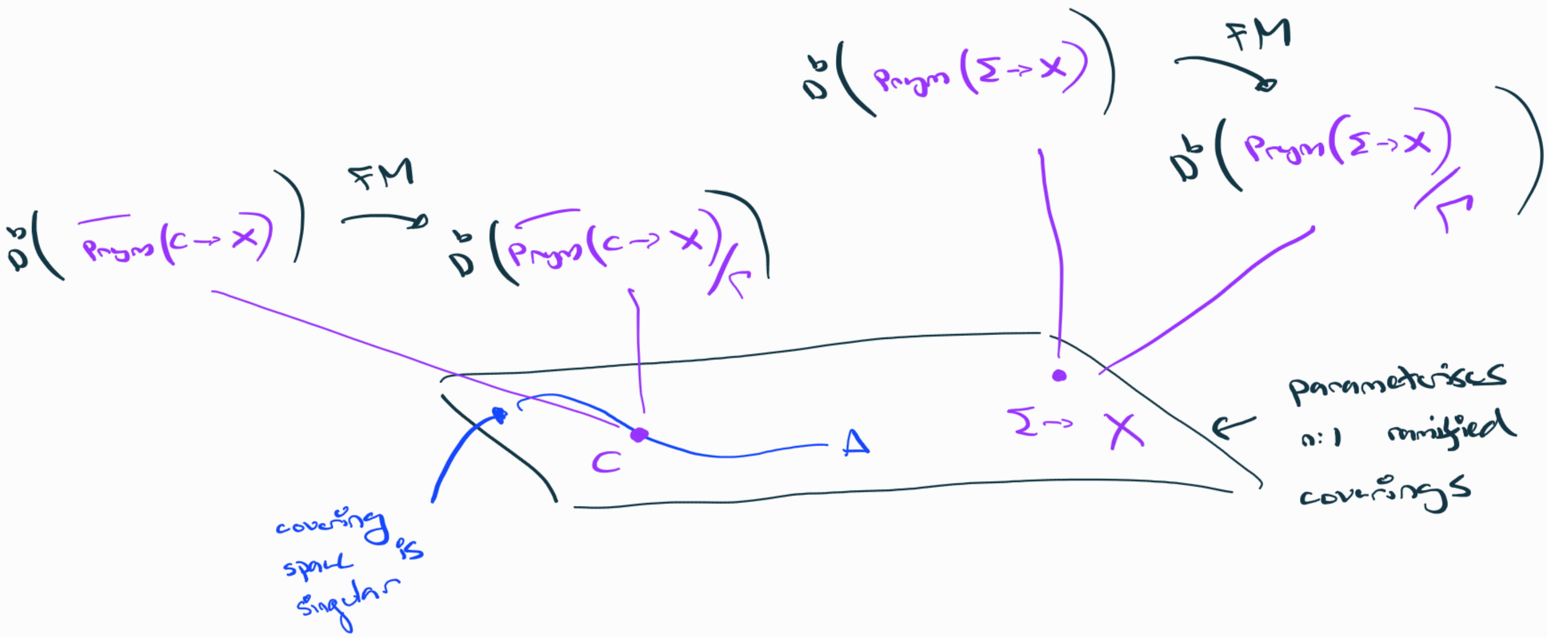
Fourier-Mukai transforms on Jacobians and Pryms





Examples of abelian varieties

Abelian variety := variety + abelian group.

$\mathrm{Pic}^0(\underline{Z}) := \{$ degree zero line bundles on $Z \}$ "fine moduli space"

projective scheme

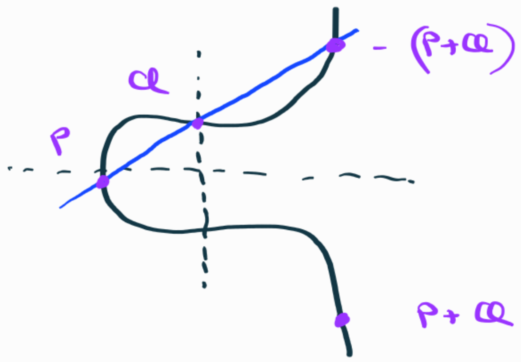
A = abelian variety, then $\hat{A} := \mathrm{Pic}^0(A)$

dual abelian variety.

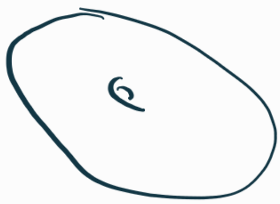
group structure ?!

Examples

Elliptic curve



$(\mathbb{A}^n/\mathbb{Z}^n, +)$



$\mathrm{Jac}(\underline{C}) := \mathrm{Pic}^0(C)$

Q: group structure ?

smooth projective curve

$\mathrm{Prym}(\beta : \underline{C \to X}) = \ker(\det \circ \beta_* : \mathrm{Jac}(C) \to \mathrm{Jac}(X))$

finite morphism of smooth projective curves

The Fourier - Mukai transforms of Mukai

$\hat{A} = \mathrm{Pic}^0(A)$ is a fine moduli space.

$\Rightarrow$ There exists a universal family:

$$\begin{array}{c} P \\ \downarrow \\ A \times A^\vee \end{array}$$

"Poincaré bundle"

[Mukai, '81] defined

$$\Phi_P : D^b(A) \longrightarrow D^b(\check{A})$$

$$\mathcal{E} \longmapsto Rp_{2*}\left(P \overset{L}{\otimes} Lp_1^* \mathcal{E}\right)$$

$$\begin{tikzcd} & A \times A^\vee \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A^\vee \end{tikzcd}$$

Prop'n [Mukai] Φ_P is an equivalence of categories

proof = combination of base change, projection formula, etc...

Examples

$A = A^\vee = \mathrm{Jac}(C)$ $\qquad D^b(\mathrm{Jac}(C)) \xrightarrow{\simeq} D^b(\mathrm{Jac}(C))$

$A = \mathrm{Prym}(\beta)$

$A^\vee = \mathrm{Prym}(\beta)/\Gamma$ $\qquad D^b(\mathrm{Prym}(\beta)) \xrightarrow{\simeq} D^b(\mathrm{Prym}(\beta)/\Gamma)$

SYZ mirror symmetry

Calabi-Yaus M and N are SYZ mirror dual if they admit dual torus fibrations

generic fibers are abelian varieties such that

$$
\begin{tikzcd}
M \arrow[dr, "h_M"'] & & N \arrow[dl, "h_N"] \\
& B &
\end{tikzcd}
$$

$$h_M^{-1}(b) \cong \widehat{h_N^{-1}(b)}$$

In derived terms:

$$\Phi : D^b(h_M^{-1}(b)) \xrightarrow{\cong} D^b(h_N^{-1}(b))$$

$$\text{skyscraper sheaves} \longmapsto \text{line bundles}$$

SYZ leads us to seek

D^b - equivalences of this form

Higgs bundles

Nigel Hitchin, 1987:

" 2D reduction of self dual Yang-Mills equations "

Solutions over smooth curve X are Higgs bundles:

$$(E, \phi)$$

vector bundle $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ sheaf morphism $\phi : E \to E \otimes K_X$

" The moduli space of all solutions turns out to be a manifold with an extremely rich geometric structure "
[Hitchin, '87]

$$M(n,d) := \left\{ \underbrace{\text{stable}}_{\text{a GIT thing}} \text{ Higgs bundles on } X \text{ of } \begin{array}{l} \text{rank} = n \\ \text{degree} = d \end{array} \right\}$$

(E, ϕ) stable if $(F, \psi) \subseteq (E, \phi) \Rightarrow \dfrac{\deg(F)}{\text{rank}(F)} \leq \dfrac{\deg(E)}{\text{rank}(E)}$

theorem [Hitchin] $M(n,d)$ is a hyperkähler manifold.

three complex structures J_i
three Kähler forms ω_i
compatible metric g

Hyperkähler $\Rightarrow$ Calabi Yau

<u>proof</u> $\Omega_1 = \omega_2 + i\omega_3$ is holomorphic - symplectic.

$\Omega_1^{\dim}$ trivialises K_X $\square$

So we can do SYZ mirror symmetry with $M(n,d)$

The fibration : "The Hitchin fibration"

$$\mathrm{char}(\phi) = \det(tI - \phi) = t^n + b_1 t^{n-1} + \dots + b_n$$

$$h : M = \{(E,\phi)\} \longrightarrow B = \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$

$$(E,\phi) \longmapsto (b_1, \dots, b_n)$$

<u>example</u> rank $= 2$

$$E = K^{1/2} \oplus K^{-1/2}$$

$$\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} : \quad \begin{tikzcd} K^{1/2} \arrow[rrrrrr, bend right=40, "1"'] & \oplus & K^{-1/2} \arrow[rr, bend right, "\omega"'] & \longrightarrow & K^{3/2} & \oplus & K^{1/2} \end{tikzcd}$$

$\omega \in H^0(X, K^2)$
"quadratic differential"

$$\mathrm{char}(\phi) = \det\left(t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} t & -\omega \\ -1 & t \end{pmatrix}$$

$$= t^2 - \omega$$

$$h\left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}\right) = (1, -\omega) \in H^0(X, K) \oplus H^0(X, K^2)$$

Spectral data

Fix $b \in B$.

Spectral curve C_b := zeros of the morphism

$$T^*X = \mathrm{tot}(K_X) \longrightarrow \mathrm{tot}(K_X^n)$$

$$x \longmapsto x^n + b_1 x^{n-1} + \dots + b_n$$

Spectral cover := ramified covering map

$$\begin{tikzcd} C_b \arrow[r, hook] \arrow[rr, bend right, "p_b"'] & \mathrm{tot}(K_X) \arrow[r, "\pi"] & X \end{tikzcd}$$

"Spectral theorem" $\quad h^{-1}(b) \cong \overline{\mathrm{Jac}}(C_b) \quad$ " $\{$ rank 1 torsion-free sheaves $\}$ "

$$= \mathrm{Jac}(C_b) \qquad (\text{when } C \text{ is smooth})$$

The map:

$$(E, \phi) \longmapsto \mathrm{coker}\left(\pi^* E \xrightarrow{\lambda \otimes 1 + \pi^* \phi} \pi^*(E \otimes K) \right)$$

example

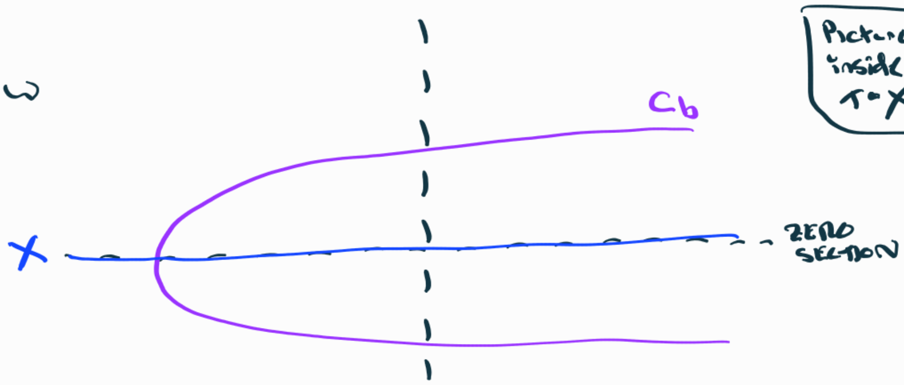
$E = K^{1/2} \oplus K^{-1/2} \qquad \phi = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}$

$b = (1, w) \in B$

C_b = zeros of $x^2 - w$

$= \{ x^2 = w \}$

Picture inside T^*X



$M(n,d)$ and SYZ

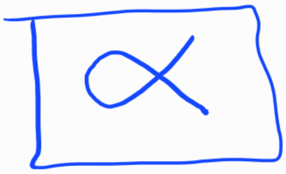
Recall: D^b- interpretation of SYZ mirror pairs

$$\begin{tikzcd} M \arrow[dr, "h_M"'] & & N \arrow[dl, "h_N"] \\ & B & \end{tikzcd}$$

was $D^b(h_M^{-1}(b)) \xrightarrow{\cong} D^b(\widehat{h_N^{-1}(b)})$

$M(n,d)$ is dual to itself!

$$\begin{tikzcd} M(n,d) \arrow[dr] & & M(n,d) \arrow[dl] \\ & B & \end{tikzcd}$$



theorem [Arinkin, 2013]

Let C be an integral and planar curve.

(1) There exists a sheaf $\overline{P}$ on $\overline{Jac}(C) \times \overline{Jac}(C)^{\wedge}$ "Poincaré sheaf"

that extends the Poincaré bundle on $Jac(C) \times \widehat{Jac(C)}$

(2) The integral functor

$$\Phi_{\overline{P}} : D^b(\overline{Jac}(C)) \longrightarrow D^b(\overline{Jac}(C)^{\wedge})$$

is an equivalence of categories.

SL - moduli space

$\mathcal{M}_{SL_n}$ = sub-moduli of $\mathcal{M}(n,0)$ defined by

$Tr(\phi) = 0$

$\det(E) \cong \mathcal{O}_X$

e.g. $\left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}\right) \in \mathcal{M}_{SL_2}$

Hitchin fibration :

$$\begin{tikzcd} \mathcal{M}_{SL_n} \arrow[r, hook] \arrow[dr, "h_{SL_n}"'] & \mathcal{M}(n,0) \arrow[d, "h"] \\ & B \end{tikzcd}$$

What is the image of h_{SL_n} ?

$char(\phi) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$

Condition on the b_i's ....

$B_{SL_n} :=$ -----

SL fibers

$b \in B_{SL_n} \rightsquigarrow p_b : C_b \longrightarrow X$ "spectral cover"

Recall $h^{-1}(b) \cong \overline{\mathrm{Jac}}(C_b)$

$h^{-1}_{SL_n}(b)$ is the subset s.t. $\det(E) \cong \mathcal{O}_X$

Recall $\overline{\mathrm{Jac}}(C_b) \xrightarrow{\cong} h^{-1}(b)$

$$\mathcal{L} \longmapsto E = p_{b*}\mathcal{L}$$

So we impose the condition $\det(p_{b*}\mathcal{L}) \cong \mathcal{O}_X$

Corollary

$$h_{SL}^{-1}(b) \cong \mathrm{Ker}\left(\det \circ p_{b*} : \overline{\mathrm{Jac}}(C_b) \longrightarrow \mathrm{Jac}(X)\right)$$

$$\text{''} \; \overline{\mathrm{Prym}}(p_b)$$

$\mathcal{M}_{SL_n}$ and SYZ

$\Gamma := \mathrm{Jac}(X)[n] = \{ L \in \mathrm{Jac}(X)$ such that $L^{\otimes n} \cong \mathcal{O}_X \}$

Γ acts on $\mathcal{M}_{SL_n}$ ....

Conjecture : [Hitchin, Hausel-Thaddeus, etc...]

The SYZ picture for $\mathcal{M}_{SL_n}$ is

$$\begin{tikzcd} \mathcal{M}_{SL_n} \arrow[dr, "h_{SL_n}"'] & & \mathcal{M}_{SL_n} /\!/ \Gamma \arrow[dl, "{[h_{SL_n}]}"] \\ & \mathcal{B}_{SL_n} & \end{tikzcd}$$

Recall : D^b- interpretation of SYZ mirror pairs

$$\begin{tikzcd} M \arrow[dr, "h_M"'] & & N \arrow[dl, "h_N"] \\ & B & \end{tikzcd}$$

was $D^b(h_M^{-1}(b)) \xrightarrow{\cong} \widehat{D^b(h_N^{-1}(b))}$

Conjecture*

$D^b(\overline{\mathrm{Prym}}(pb : C_b \to X)) \longrightarrow D^b(\overline{\mathrm{Prym}}(pb : C_b \to X) /\!/ \Gamma)$

we discovered that this conjecture is wrong!

Instead:

__theorem__ [Franco - H - Ruano, 22]:

C = projective, reduced, connected, locally planar

$\beta : C \to X$ flat + ramified $n:1$ cover

$\overline{R}$ = pullback of $\overline{P}$ along $\overline{\mathrm{Prym}}(\beta) \times \overline{\mathrm{Prym}}(\beta) \hookrightarrow \overline{\mathrm{Jac}}(C) \times \overline{\mathrm{Jac}}(C)$

Then the integral functor

$$\Phi_{\overline{R}} : D^b\left(\overline{\mathrm{Prym}}\left(\beta : C \to X\right)\right) \longrightarrow \underbrace{D^b\left(\overline{\mathrm{Prym}}\left(\beta : C \to X\right), \Gamma\right)}_{\Gamma\text{-equivariant derived category}}$$

is an equivalence of categories.

Γ acts free $\Rightarrow$ conjecture true.

But in general there are stabilizers to deal with.

Stacks

Fascinatingly : $D^b\left(\left[\overline{\mathrm{Prym}}(\beta)/\Gamma\right]\right) \cong D^b\left(\overline{\mathrm{Prym}}(\beta), \Gamma\right)$

stacky quotient

So $\overline{\mathrm{Prym}}(\beta)$ and $\left[\overline{\mathrm{Prym}}(\beta)/\Gamma\right]$ are derived equivalent.

Stacky SYZ ?

Not really, just the Geometric Langlands Programme!

<u>Conjecture</u> [Donagi-Pantev, 2006]

$$D^b(\mathrm{Higgs}_{SL_n}) \xrightarrow{\cong} D^b\left(\left[\mathrm{Higgs}_{SL_n} /\!/ \Gamma\right]\right)$$