Guide to Orlov's theorem

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Punchline

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Some consequences:

- $[\mathbb{A}^n/\mathbb{Z}_d]$ has a d-torsion orbifold line bundle
  $\implies \exists T \in \mathrm{Aut}(D^b(X))$ with $T^d = [2]$, for Calabi–Yau X.

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- The canonical bundle of $[\mathbb{A}^n/\mathbb{Z}_d]$ is d-torsion $\implies$ Kuznetsov components are fractional Calabi–Yau (for Fano hypersurfaces / complete intersections)

Proof structure

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$$\begin{array}{c} (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) \\ \Big\updownarrow \text{Knörrer periodicity} \\ X \end{array}$$

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Hypersurface $X \subset \mathbb{P}^{n-1}$, $f \in H^0(\mathcal{O}(d))$.

$$\begin{tikzcd}
 & ([\mathbb{C}^{n+1}/(\mathbb{C}^*)_{1,\dots,1,-d}], fp) \arrow[dl, "\{x\neq 0\}"', "\mathrm{GIT}"] \\
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Two steps:

1. KP - easy geometric functor (push-pull).
2. VGIT - mildly computationally annoying functor (iterate cones to move something into a window).

Tower of Babel

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- *Open sector* of B-model $\rightsquigarrow$ dg-category of B-branes = matrix factorization category
- *Closed sector* of B-model $\rightsquigarrow$ $HH_\bullet$ of B-brane dg-category (HKR isomorphism $\rightsquigarrow = H^\bullet(\Omega_X^\bullet, dW) = \mathrm{Jac}(W)$)

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Physics jargon:

- *Landau–Ginzburg A-model* $\rightsquigarrow$ (X, W); manifold + global function (symplectic Lefschetz fibration)
- *Superpotential* $\rightsquigarrow$ global function W
- *A-branes* $\rightsquigarrow$ thimbles / vanishing cycles
- *Open sector* of A-model $\rightsquigarrow$ A_∞-category of A-branes $=$ Fukaya–Seidel category
- *Closed sector* of A-model $\rightsquigarrow$ $HH_\bullet$ of A-brane A_∞-category (GPS, Abouzaid–Seidel, ... $\rightsquigarrow$ $=$ $SH^\bullet$(Milnor fiber) $=$ $\mathrm{Jac}(W)$)

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Curved dg-algebra: (A, d, w) with $|w| = 2$, $d^2 = [w, -]$ and $dw = 0$.
Curved dg-module: (M, d) with $d^2(-) = w(-)$.
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Graded algebra A with $|w| = n$ central.
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X variety with G action, $w \in \Gamma(X, L)$ for G-equivariant line bundle L.
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Orlov: $X = \operatorname{Spec} A$, $G = \mathbb{G}_m$, $L = \mathcal{O}(\chi^n_{\mathbb{G}_m})$ [like $\operatorname{Proj} A$ with $\mathcal{O}(n)$]

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Definition (Segal 2011)

Grading on $\mathcal{O}_X$: $\mathbb{C}^*$-action on X, $w \in \Gamma(\mathcal{O}_X)$ weight 2.
Sheaves of curved dg-modules over $(\mathcal{O}_X, w)$:

- graded module $=$ $\mathbb{C}^*$-equivariant sheaf,
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Fact

This is a special case of BFK definition with $G = \mathbb{C}^*$, $L = \mathcal{O}(\chi^2_{\mathbb{C}^*})$.
Conversely, BFK definition is equivalent to taking $X = [U(L)/G]$ here.

Derived categories: Objects

E.g. $D^b(\mathbb{P}^2)$, chain complexes, $d^2 = 0$:

$$\left(\mathcal{O}(-2)[2] \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}(-1)^2[1] \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O} \right) \simeq \mathcal{O}/(x,y)$$

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Totalizations of complexes of complexes (e.g. cones):

$$\mathrm{Tot}\left(\begin{array}{ccc} \mathcal{O}(-2)[2] & \xrightarrow{x} & \mathcal{O}(-1)[1] \\ \downarrow y & & \downarrow y \\ \mathcal{O}(-1)[1] & \xrightarrow{x} & \mathcal{O} \end{array} \right) \simeq \left(\mathcal{O}/y(-1)[1] \xrightarrow{x} \mathcal{O}/y \right) \simeq \mathcal{O}/(x,y)$$

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Definition (Orlov 2011 "... nonaffine LG-models")

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Example (Koszul resolutions)

Exact sequence:

$$\begin{tikzcd} \mathcal{O}(3) \arrow[r, "p"] \arrow[d, shift left, "xyzp"] & \mathcal{O} \arrow[r, "1"] \arrow[d, shift left, "xyz"] & \mathcal{O}/p. \\ \mathcal{O}(3)[1] \arrow[r, "1"] \arrow[u, shift left, "1"] & \mathcal{O}(3)[1] \arrow[u, shift left, "p"] & \end{tikzcd}$$

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Matrix factorizations: derived Homs

Fact (Koszul resolutions)

In $D^b(X)$, $Z \subset X$ complete intersection, $\mathrm{RHom}(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{R}\Gamma(\wedge^\bullet N_{Z/X}[-1])$.

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where $\partial_x \lrcorner dW = yz = 0$, $\partial_y \lrcorner dW = xz = 0$.
For $\mathrm{MF}(\mathbb{A}^3, yz^2)$: $\partial_x \lrcorner dW = 0$ but $\partial_y \lrcorner dW = z^2$, so

$$\mathrm{RHom}(\mathcal{O}/(x,y), \mathcal{O}/(x,y)) \simeq (\mathbb{C}[z]/z^2)\langle \partial_x \rangle.$$

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Recall: $X \subset \mathbb{P}^{n-1}$ hypersurface, $f \in H^0(\mathcal{O}(d))$.

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Sketch of proof.

$$\begin{tikzcd} \mathcal{O}_X(-d) \arrow[r, hook, "j"] \arrow[d, "\pi"] & (\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp) \arrow[d, "\pi"] \\ X \arrow[r, hook, "j"] & \mathbb{P}^{n-1}. \end{tikzcd}$$

Compute that $j_*\pi^* : D^b(X) \to \mathrm{MF}(\mathcal{O}_{\mathbb{P}^{n-1}}(-d), fp)$ is an equivalence. $\square$

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□

VGIT and windows

Recall:

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\begin{tikzcd}[column sep=small]
 & ([\mathbb{C}^{n+1}/\mathbb{C}^*_{1,\dots,1,-d}], fp) \arrow[dl, "j_L^*"', "\{x\neq 0\}"] \arrow[dr, "j_R^*", "\{p\neq 0\}"'] & \\
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Definition

The *window category* for the interval $[k : k+l-1]$ is

$$\mathcal{W}_{k,l} = \text{``}\langle \mathcal{O}(k), \mathcal{O}(k+1), \dots, \mathcal{O}(k+l-1)\rangle\text{''} \subset \mathrm{MF}([\mathbb{C}^{n+1}/\mathbb{C}^*], fp).$$

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Corollary

- *If* $d = n$, $D^b(X) \simeq \mathrm{MF}([\mathbb{A}^n/\mathbb{Z}_d])$.
- *If* $d < n$, $\mathrm{MF}([\mathbb{A}^n/\mathbb{Z}_d]) \subset D^b(X)$.
- *If* $d > n$, $D^b(X) \subset \mathrm{MF}([\mathbb{A}^n/\mathbb{Z}_d])$.

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The normal bundle of the unstable locus tells you the window size. □

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So $\mathcal{O}_p \mapsto \mathcal{O}/(x, y) \in \mathrm{MF}([\mathbb{A}^3/\mathbb{Z}_3], y^2z - x^3 - xz^2)$.

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\mathcal{O}(-3)[1] \arrow[r, shift left, "(-z\ y\ -x)^t"] \arrow[d, shift right, "f"'] & \mathcal{O}(-2)^3 \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "\cdots"] \arrow[dl, bend left, "(xz\ yz\ x^2)"] & \mathcal{O}(-1)^3[-1] \arrow[l, shift left, "\cdots"] \arrow[r, shift left, "(x\ y\ z)"] & \mathcal{O}[-2] \arrow[l, shift left, "\cdots"] \\
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Then restrict this to $\{p \neq 0\}$.

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As expected: $R\Gamma(\mathcal{O}_E) = \mathbb{C} \oplus \mathbb{C}[-1]$.

Thanks for listening!