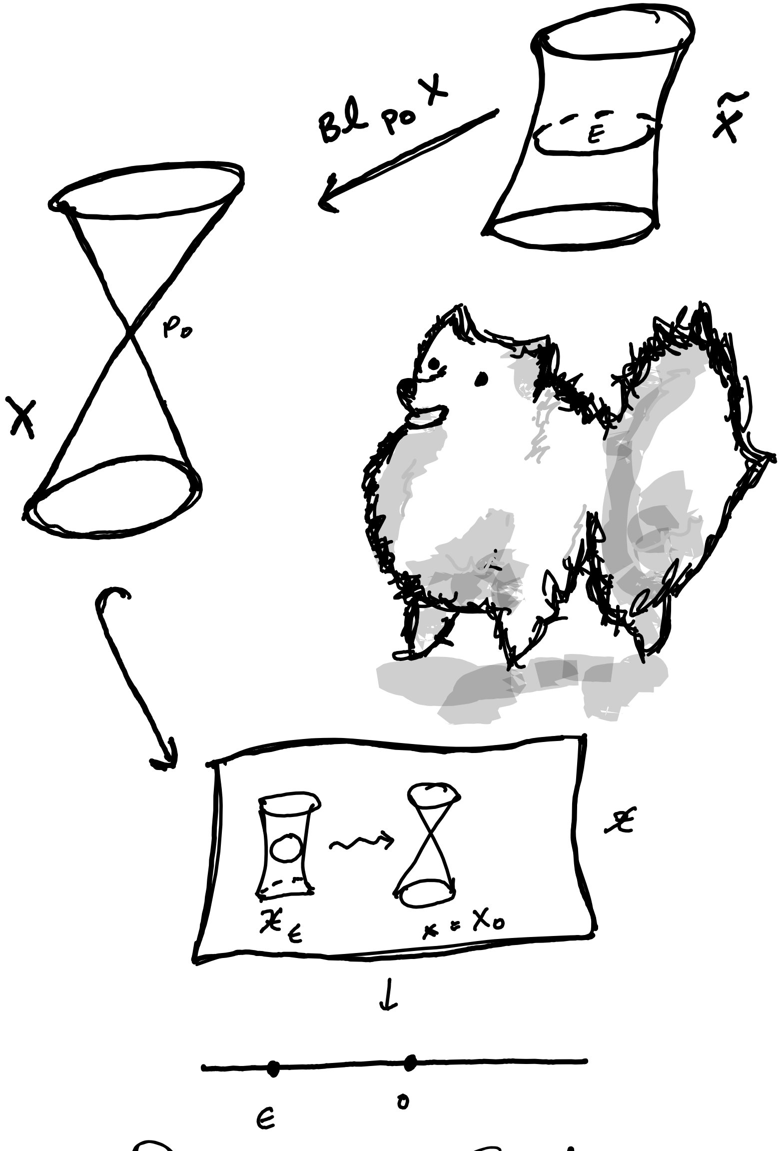


Derived Categories of Nodal Varieties

~ DOGS 26109 ~

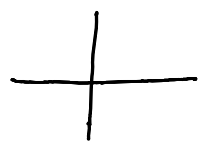
1. Nodal Varieties

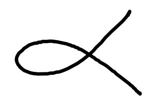
- Dimension 0:

  Spec $k[\epsilon]/\epsilon^2$

- Dimension 1:

  $\{xy = 0\}$ $\qquad$ $\{y^2 = x(x-1)^2\}$

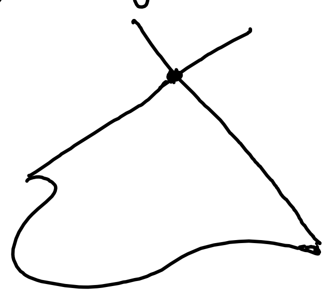




- Dimension n: take a cone over a smooth quadric of dim. $n-1$

  e.g. $\{xy = z^2\} \subset \mathbb{P}^2$

  Take a cone $\{xy = z^2\} \subset \mathbb{P}^3_{x:y:z:w}$ has singularity at $[0:0:0:1]$



- We'll (mostly) consider projective examples

2. $\mathbb{P}^\infty$-objects.

- we notice that:

  $D^b(\operatorname{Spec} k[\epsilon]/\epsilon^2)$

  $\simeq \langle k \rangle$

  where $k = \mathcal{O}/\epsilon$

  $\mathcal{O} = k[\epsilon]/\epsilon^2$

- What is $\operatorname{Ext}^\bullet(k,k)$?
- Free resolution:

$$0 \leftarrow k \xleftarrow{\bmod \epsilon} \mathcal{O} \xleftarrow{\cdot\epsilon} \mathcal{O} \xleftarrow{\cdot\epsilon} \mathcal{O} \xleftarrow{\cdot\epsilon} \cdots$$

Apply $R\mathrm{Hom}(-,k)$:

$$k \xrightarrow{0} k \xrightarrow{k} k \longrightarrow \cdots$$

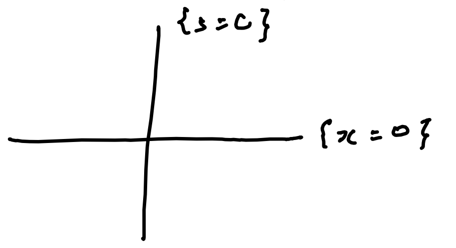
- so $\operatorname{Ext}^i(k,k) \cong k$ for $i \geq 0$

  $\Rightarrow \operatorname{Ext}^\bullet(k,k) \cong k[\theta]$

  where $|\theta| = 1$

- Now consider

  $\{xs = 0\} \subset \mathbb{P}^1_{x:y} \times \mathbb{P}^1_{s:t}$



and think about the skyscraper sheaf $\mathcal{O}/s =: P$

- $\operatorname{Ext}^\bullet(P,P)$: free resolution

$$0 \leftarrow P \leftarrow \mathcal{O} \xleftarrow{\cdot s} \mathcal{O}(-1,0) \xleftarrow{\cdot x} \mathcal{O}(-1,-1)$$
$$\xleftarrow{\cdot s} \mathcal{O}(-2,-1) \xleftarrow{\cdot x} \cdots$$

- Apply $\mathrm{Hom}(-,P)$:

$$P \xrightarrow{0} P \xrightarrow{\cdot x} P(1) \xrightarrow{\cdot 0} P(1) \longrightarrow \cdots$$

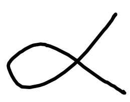
$\mathcal{E}xt^i$: P 0 $\mathcal{O}_{pt}$ 0 $\cdots$

Ext^i: k 0 k 0 $\cdots$

So $\operatorname{Ext}^\bullet(P,P) = k[\theta]$

$|\theta| = 2$

- We call P a $\mathbb{P}^{\infty,q}$ object where $q = \deg\theta$
- $\mathbb{P}^\infty$ objects don't always exist, e.g. consider



so we have global obstructions

- $\mathbb{P}^{\infty,1}$ objects appear on nodal varieties of even dim.
- $\mathbb{P}^{\infty,2}$ objects appear on nodal varieties of odd dim.

3. SODs / Absorption.

- Consider $\underbrace{\{xs = 0\}}_{=C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ and the contraction map

$$f : C \longrightarrow \mathbb{P}^1_{s:t}$$

- The fact that $f_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1}$ means we have an SOD:

$$D^b(C) = \langle \ker f_*, f^* D^b(\mathbb{P}^1) \rangle$$

and intuitively,

$$\ker f_* = \langle \mathcal{O}_{\{s=0\}}(-1) \rangle$$

which we already know is a $\mathbb{P}^{\infty,2}$ object.

- Call this absorption of singularities since $D^b(\mathbb{P}^1)$ is derived category of smooth & projective variety.

4. Deformation Absorption.

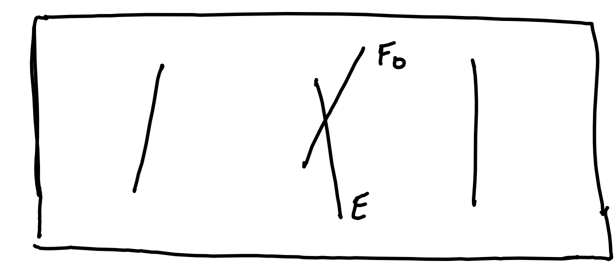
- Consider

$$Bl_{\langle x, u\rangle} \mathbb{P}^1_{x:y} \times \mathbb{P}^1_{u:v}$$

$$= \{ xs = ut \}$$

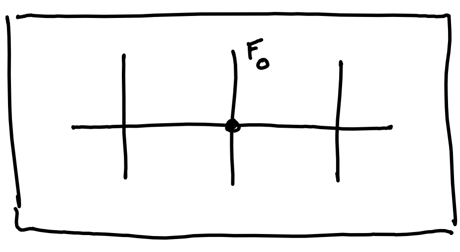
Think of this as a surface over $\mathbb{P}^1_{u:v}$

- When $u \neq 0$, we just have $\mathbb{P}^1_{x:y}$
- When $u = 0$, we get $\{xs = 0\}$



$= \mathfrak{X}$

$\downarrow$ Blow up at $\{x = u = 0\}$



$\mathbb{P}^1_{u:v} \times \mathbb{P}^1_{x:y}$

- $\mathfrak{X}$ is a smoothing of our nodal curve C

- By Orlov's blow up formula:

$$D^b(\mathfrak{X}) = \langle i_* \mathcal{O}_E(-1),\ D^b(\mathbb{P}^1 \times \mathbb{P}^1) \rangle$$

1. Firstly, $i_* \mathcal{O}_E(-1)$ is the push-forward of our $\mathbb{P}^\infty$ object on C to $\mathfrak{X}$.

2. Secondly, if we "base change" the SOD to a generic fiber, we just get $D^b(\mathbb{P}^1)$, because $i_* \mathcal{O}_E(-1)$ is supported on the central fiber.

   - The $\mathbb{P}^{\infty,2}$ object vanishes as we smooth the curve.

- Call this "deformation absorption."
- This happens with any smoothing:

Proof:

Let $i : X \hookrightarrow \mathfrak{X}$ and suppose

$$D^b(X) = \langle P, D \rangle \text{ is an SOD}$$

where P is $\mathbb{P}^{\infty,2}$.

Then consider $i_* P$ on $\mathfrak{X}$, compute $\mathrm{Ext}^\bullet(i_* P, i_* P)$:

$$\mathrm{Ext}^\bullet(i_* P, i_* P)$$

$$= \mathrm{Ext}^\bullet(i^* i_* P, P) \quad \text{(adjoints)}$$

We have a standard distinguished triangle:

$$P \otimes \mathcal{O}_X(-X)[1] \to i^* i_* P \to P$$

we know $\mathcal{O}_X(-X) = \mathcal{N}^\vee_{X/\mathfrak{X}} = \mathcal{O}$ (X lives in a family)

so:

$$i^* i_* P \to P \to P[2]$$

the map $P \to P[2]$ is non-zero, since $i^* i_* P$ is perfect, so it must be

$$P \xrightarrow{\cdot\theta} P[2].$$

Now apply $\mathrm{Hom}(-, P)$:

$0 \to \mathrm{Ext}^{-2}(P, P) \to \mathrm{Hom}(P, P)$

$\to \mathrm{Hom}(i^* i_* P, P) \to \mathrm{Ext}^{-1}(P, P)$

$\to \mathrm{Ext}^1(P, P) \to \mathrm{Ext}^1(i^* i_* P, P)$

$\to \mathrm{Ext}^0(P, P) \to \mathrm{Ext}^2(P, P)$

$\to \mathrm{Ext}^2(i^* i_* P, P) \to \dots$

gives us

$0 \to \mathrm{Hom}(P, P)$

$\xrightarrow{\sim} \mathrm{Hom}(i^* i_* P, P) \to 0$

$\to 0 \to \overset{=0}{\mathrm{Ext}^1(i^* i_* P, P)}$

$\to \mathrm{Ext}^0(P, P) \xrightarrow[\sim]{\theta} \mathrm{Ext}^2(P, P)$

$\to \underset{0}{\underset{\shortparallel}{\mathrm{Ext}^2(i^* i_* P, P)}} \to \dots$

$\Rightarrow \mathrm{Ext}^\bullet(i^* i_* P, P) = k$

$\Rightarrow i_* P$ is exceptional.

So $D^b(\mathfrak{X}) = \langle i_* P, {}^\perp(i_* P) \rangle$ is an SOD.

$\square$