Window categories, tilting objects and the derived equivalence

0.1 $Y = V(f) \subset \mathbb{P}^5$

$$\mathrm{Hom}(\mathbb{C}^6, \mathcal{U}^\vee) \oplus \mathrm{Sym}^3 \mathcal{U}^\vee$$

$$\mathcal{X} = [V/G] \hookrightarrow \mathrm{Hom}(\mathbb{C}^6, \mathcal{U}^\vee) \oplus \mathrm{Sym}^3 \mathcal{U}^\vee$$

$$\mathcal{X} \supseteq \mathcal{G} \hookrightarrow GL\,\mathcal{U}$$

$$X_- = \mathrm{Tot}_{Gr(2,6)}(\mathrm{Sym}^3 \mathcal{U}) \xrightarrow{\sim} \mathcal{G} \xleftarrow{\sim} X_+ \supseteq X_+^0$$

$$D^b(F_Y) \cong MF(X_-, \underset{\underset{f}{\wr\!\wr}}{W}) \cdots\cdots MF(X_+^0, W) \cong \mathrm{Sym}^2 A_Y$$

Today: $D^b(X_-) \simeq D^b(X_+)$

$$D^b(X_-) \simeq D^b(A) \simeq D^b(X_+)$$

$$\Downarrow$$

$$MF(X_-, W) \simeq MF(X_+, W) \simeq MF(X_+^0, W)$$

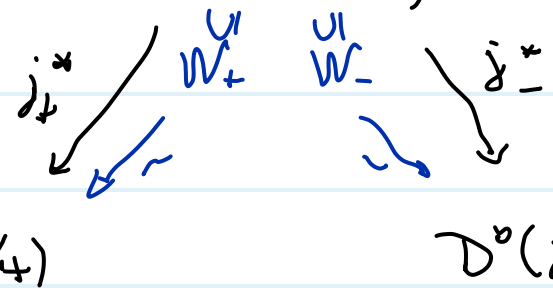
$$MF(X_-, W) \simeq MF(A, W) \simeq MF(X_+, W)$$

$$\mathrm{Crit}(W) \subset X_+^0$$

1. Window categories

1] $\mathbb{C}^* \curvearrowright \mathbb{C}^n$ $(v_1, \dots, v_n)$ weights $T^+ = \sum_{v_i > 0} v_i$

$T^- = \sum_{v_i < 0} v_i$



$$D^b([\mathbb{C}^n/\mathbb{C}^*]) = \langle \mathcal{O}(i) \rangle_{i \in \mathbb{Z}}$$

$$W_{[0,T^+)} := \langle \mathcal{O}, \dots, \mathcal{O}(T^+ - 1) \rangle \quad (W_{(-T^-,0]} \text{ analogously})$$

Thm: $j_+^* : W_{[0,T^+)} \longrightarrow D^b(X_+)$ is an equivalence.

Sketch:
- $H^i(\mathbb{C}^n, \mathcal{O}(i-j)) \longrightarrow H^i(X_+, \mathcal{O}(i-j))$ + Mayer Viet + Cohomology on X_+^{ns} } ff
- $D^b(*) \longrightarrow D^b(X_+)$ is essent. surj. and you can obtain all $\underline{\mathcal{O}(i)}$ from $\underline{W}$ } $\Rightarrow$ ess. surj.

$\qquad\qquad\qquad

1) Flips and Flops

$\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+m}$, $(1, \dots, 1, -1, \dots, -1)$ $T^+ - T^- = 0$ (CY).

$$\begin{tikzcd}
W_{[0,n)} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr"] & D^b(\mathcal{X}) \arrow[dl] \arrow[dr] & W_{[0,m)} \arrow[d, phantom, "\wr"] \\
D^b(\mathrm{Tot}_{\mathbb{P}^{n-1}} \mathcal{O}(-1)^m) & & D^b(\mathrm{Tot}_{\mathbb{P}^{m-1}} \mathcal{O}(-1)^n)
\end{tikzcd}$$

2) Orbifold

$\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+1}$ $(1, \dots, 1, -n)$ CY $(T^+ - T^- = 0)$

$$\begin{tikzcd}
& D^b(\mathcal{X}) \arrow[dl] \arrow[dr] & \\
D^b(\mathrm{Tot}_{\mathbb{P}^{n-1}} \mathcal{O}(-n)) \arrow[r, phantom, "\sim"] \arrow[dr, phantom, "\sim"] & W_{[0,k)} \arrow[r, phantom, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & D^b([\mathbb{C}^n/\mathbb{Z}_n]) \arrow[dl, phantom, "\sim"] \\
& W_{[k,k+n)} &
\end{tikzcd}$$

General situation: Graded restriction subcategory.

$$\mathcal{G} \subseteq D^b(\mathcal{X}) \longrightarrow D^b(X_+)$$

$\mathcal{X} = [V/G]$ $\rightsquigarrow$ (fix $+$)

(In our set up G has only 2 characters $+$ and $-$)

• λ_i a 1-PS of G

Stratification of X_+^{us}

→ Z_i fixed points of λ_i.

and $Y_i = \{x \in V \text{ s.t. } \lim_{t\to 0} \lambda_i(t)\cdot x \in Z_i\}$

$$S_i = G\cdot Y_i \rightsquigarrow \bigcup_{\lambda_i} S_i = X_+^{us}$$

Let $F^\cdot \in D^b(X)$, take $H^\cdot(F^\cdot_{|Z_i})$ then it admits a decomposition in weights

We declare $F^\cdot \in \mathcal{G}$ if these weights lie in $[0, \eta_i)$

⇒ η_i = $\sum$ positive weights of λ_i − $\sum$ positive weights of g

= $\sum$ weight of $\det(N^\vee_{S_i/V|Z_i})$

Thm $\mathcal{G}_+ \simeq D^b(X_+)$, $\mathcal{G}_- \simeq D^b(X_-)$

2.1 Tilting objects $D^b(X) = \langle E_1, \dots, E_n \rangle$

$T = E_1 \oplus \dots \oplus E_n$ is a generator, $\mathrm{Hom}(T, F) = 0 \Rightarrow F = 0$

$R\mathrm{Hom}(T, -) : D^b(X) \xrightarrow{\Psi} D^b(A) : - \otimes_A T$

$A = \mathrm{End}(T)$ ⚠ Assuming that $\mathrm{End}(T)$ is ---------- an algebra.

For this we need

$$\mathrm{Ext}^i(T,T)=0 \text{ for } i>0$$

this happens for strong f.e.c.

$$\mathrm{Ext}^k(\mathcal{E}_i,\mathcal{E}_j)=0,\ k>0.$$

Def T tilting if T generates $D^b(X)$ and

$$\mathrm{Ext}^i(T,T)=0 \text{ for } i>0$$

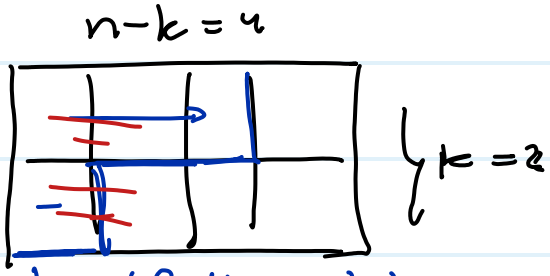
Th If T is tilting then Ψ is an equivalence.

Ex: 1) $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$

is

$D^b(\mathrm{End}(\mathcal{O}\oplus\cdots\oplus\mathcal{O}(n)))$

2) Grassmannians $Gr(2,6) = \left[\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^6)/GL_2\right]$



$\lambda=(3,1)$ $\{\lambda=(a,b)\ 0\le b\le a\le 4\}$

- Schur functor $\mathbb{S}_\lambda(\mathcal{U}^\vee)$:  k $\Lambda^k \mathcal{U}^\vee$

 $\mathrm{Sym}^\delta \mathcal{U}^\vee$

$D^b(Gr(2,6)) \simeq \langle \mathbb{S}_\lambda(\mathcal{U}^\vee) \rangle_{\lambda \subset}$  s.f.e.c.

$\mathbb{S}_{(a,b)}(\mathcal{U}^\vee) = \mathrm{Sym}^{a-b} \mathcal{U}^\vee \otimes (\det \mathcal{U}^\vee)^b$ 15 objects for $Gr(2,6)$

3.] Kembai-Segal

$$\begin{tikzcd}
 & D^b(\mathcal{X}) \ni \mathcal{T}_{(2,-2)} \arrow[dl] \arrow[dr] \arrow[dl, dashed] \arrow[dr, dashed] & \\
D^b(\mathrm{Tot}_{Gr(2,6)} \mathrm{Sym}^3 \mathcal{U}) & & D^b(X_+)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X} = [\mathrm{Hom}(\mathbb{C}^6, \mathcal{U}^\vee) \oplus \mathrm{Sym}^3 \mathcal{U}^\vee / GL_2 \mathcal{U}] \arrow[d, "q"] \\
Gr(2,6) = [\mathrm{Hom}(\mathbb{C}^6, \mathcal{U}^\vee) / GL_2 \mathcal{U}]
\end{tikzcd}$$

$\mathcal{T}_{(a,b)} = q^* \mathbb{S}_{(a,b)}(\mathcal{U}^\vee)$

$D^b(X) = \langle E_1, \dots, E_m, P \rangle$

$K_0(X) = K_0(E_1) \oplus \cdots \oplus K_0(E_m) \oplus K_0(P) \to 0$ $\quad \cong \mathbb{Z}^{\oplus m}$ $\quad \operatorname{rk} K_0(X) = \chi(X)$

We have $\chi(X_-) = \chi(X_+)$

- Check if $T_{(a,b)}$ lie in G_- and G_+ ⇝ (picture of ∇_- and ∇_+ below)

X_-

→ $\mathcal{E}_- = \bigoplus_{\nabla_-} T_{(a,b)}$ $\qquad E_- := \mathcal{E}_{-|X_-}$

Thm E_- is a tilting object. $\Rightarrow D^b(X_+) \simeq D^b(\operatorname{End} E_-)$.
$\qquad\qquad\qquad

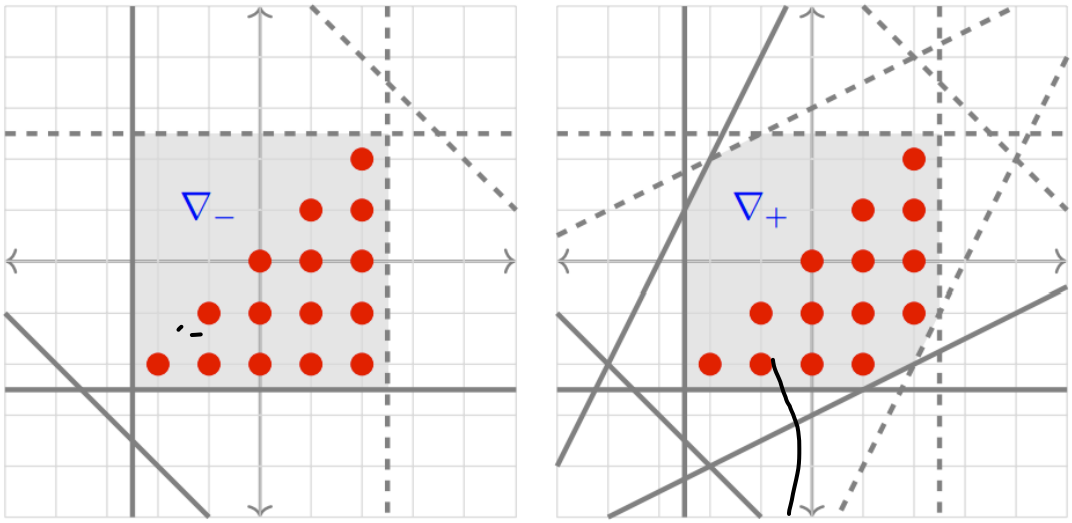


FIGURE 1. The red dots are the dominant weights corresponding to vector bundles that satisfy the grade restriction rules for $\mathcal{G}_-$ (left) and $\mathcal{G}_+$ (right).

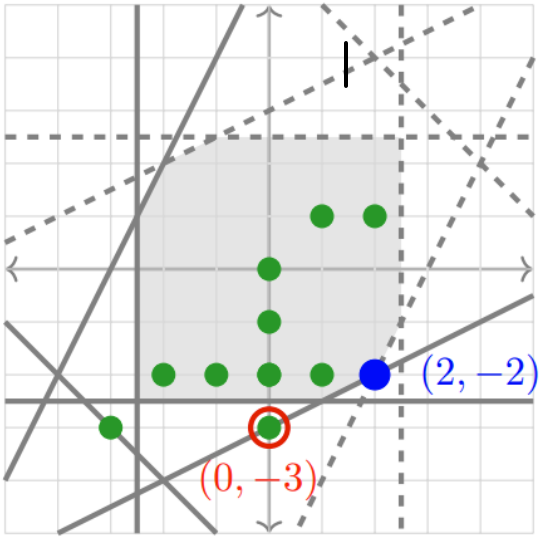


FIGURE 2. The bundles appearing in the free resolution of $\mathcal{K}$ (in green) together with $\mathcal{T}_{(2,-2)}$ (in blue). The final term of the resolution is circled in red. The grade restriction rules for $\mathcal{G}_+$ are also shown.