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Homological Mirror symmetry in Toric GIT

1. Homological Mirror Symmetry Conjecture blueprint

[Kontsevich, 1994]

Given a complex geometry X, there exists a symplectic geometry $\overline{X}$ such that

$$D^b(X) \cong \mathrm{Fuk}(\overline{X})$$

Warning: This is just a blueprint.

We have a families of mirrors

$$\pi_1(\overline{\mathcal{M}}_{cs}) \curvearrowright \mathrm{Fuk}(\overline{X})$$

Brane monodromy conjecture

$\pi_1(\overline{\mathcal{M}}_{cs})$ acts on $D^b(X)$ via monodromy

Examples of HMS

1. [Calabi-Yau, non compact]

$\mathbb{C}^* \leftrightarrow \mathbb{C}^*$ $\quad D^b(\mathbb{C}^*) \cong W\mathrm{Fuk}(\mathbb{C}^*)$

$\mathcal{O} \overset{?}{\leftrightarrow}$

2. [Fano]

$\mathbb{P}^1 \leftrightarrow (\mathbb{C}^*, W = z + \frac{q}{z}, \mathbb{C}^* \to \mathbb{C})$ $\quad D^b(\mathbb{P}^1) \cong FS(\mathbb{C}^*, W)$

Landau Ginzburg Model

superpotential $q \in \mathbb{C}^*$

$D^b(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle_{SOD}$ Beilinson's

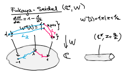
$\mathcal{O} \leftrightarrow L_1$

$\mathcal{O}(1) \leftrightarrow L_2$

Fukaya-Seidel $(\mathbb{C}^*, W)$

$\frac{dW}{dz} = 1 - \frac{1}{z^2}$

$W^{-1}(\lambda) = \{z \mid z + \frac{q}{z} = \lambda\}$



general takeaway: compactifying ⇔ adding a superpotential

Goal • Explain HMS for toric Calabi-Yau geometric invariant theory

Why? • toric geometry is combinatorial, so we can compute stuff

• GIT gives us lots of autoequivalences with which $\pi_1(\overline{\mathcal{M}}_{cs})$ can act.

Toric Geometric Invariant Theory

$T \curvearrowright V$, $T \cong (\mathbb{C}^*)^r$, $V \cong \mathbb{C}^n$ $\quad (\lambda_1,\dots,\lambda_r)\cdot(z_1,\dots,z_n) = (\lambda_1^{q_{11}}\cdots\lambda_r^{q_{r1}} z_1, \dots, \lambda_1^{q_{1n}}\cdots\lambda_r^{q_{rn}} z_n)$ $\quad q_{ij} \in \mathbb{Z}$

All of this data is packaged in a $r \times n$ integer matrix $Q = (q_{ij})$, the weight matrix

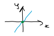
We call the columns the weights

- **GIT**: want a variety that parametrises the orbits of T
- Problem: Because the action is not free, we can't naively take the quotient

GIT Example 1 $\mathbb{C}^*_\lambda \curvearrowright \mathbb{C}^2_{x,y}$

$\lambda\cdot(x,y) = (\lambda x, \lambda y)$ $\quad Q = (1 \quad 1)$

$\mathbb{C}^*$-orbits: $\mathbb{C}^2_{x,y}$



⚠ The topological quotient is not Hausdorff!

GIT tells us to remove $(0,0)$

$$\mathbb{C}^2 /\!\!/ \mathbb{C}^* = \mathbb{C}^2 \setminus \{(0,0)\} / \mathbb{C}^* = \mathbb{P}^1$$

GIT tells us an unstable locus to remove so that the naive 'quotient is good'

Important fact: GIT quotients are **NOT** unique.

GIT quotients depend on a choice of stability condition

This is just a choice of character $\theta : (\mathbb{C}^*)^r \to \mathbb{C}^*$

$\theta(\lambda_1,\dots,\lambda_r) = \lambda_1^{\theta_1}\cdots\lambda_r^{\theta_r}$

$(\theta_1,\dots,\theta_r) \in \mathbb{Z}^r$

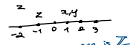
$\Rightarrow$ stability conditions live in a rank r lattice $L \cong \mathbb{Z}^r$, called the character lattice

GIT Example 2 $\mathbb{C}^*_\lambda \curvearrowright \mathbb{C}^3_{x,y,z}$

$\lambda\cdot(x,y,z) = (\lambda x, \lambda y, \lambda^{-1} z)$

$$Q = \begin{pmatrix} 1 & 1 & -1 \\ x & y & z \end{pmatrix}$$

$L \cong \mathbb{Z}$



The stability conditions space is $\mathbb{Z}$

- $\theta > 0$, $Z_+ = V(\mathcal{I}_+) = V(x,y) = \{x=y=0\}$
  $\mathbb{C}^3 /\!\!/_\theta \mathbb{C}^* = \mathbb{C}^3 \setminus \{x=y=0\} / \mathbb{C}^* = \mathrm{Tot}\, \mathcal{O}_z(-1)_{\mathbb{P}^1_{x,y}}$
- $\theta = 0$, $Z_0 = V(\mathcal{I}_0) =$
  $\mathbb{C}^3 /\!\!/_0 \mathbb{C}^* = \mathbb{C}^3 \setminus \quad / \mathbb{C}^* = \mathbb{A}^2_{x,y} \setminus \{0\}$
- $\theta < 0$, $Z_- = V(\mathcal{I}_-) = V(z) = \{z=0\}$
  $\mathbb{C}^3 /\!\!/_\theta \mathbb{C}^* = \mathbb{C}^3 \setminus \{z=0\} / \mathbb{C}^* = \mathbb{A}^2_{x,y}$

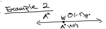
Remark $\theta = 0$ made us throw away more than we needed to $\Rightarrow$ The top dimensional cones give the best GIT quotients. From now on these will be what we mean by GIT quotient.

Important observation:

- We see a wall-and-chamber decomposition on the stability conditions L.

We get a toric fan, where stability conditions lying in the interior of the same cone determine the same unstable locus.

e.g. Example 2



Wall Crossing Formula **[The derived cat of GIT quotients, DHL]**

Let X & X' be two toric GIT quotients which are separated by a wall W

There is (ignoring ordering) an SOD

$$D^b(X) = \langle D^b(X'), \underbrace{D^b(Z), \dots, D^b(Z)}_{\kappa\text{-times}} \rangle_{SOD}$$

where Z is a strictly lower dim variety and κ is given by a simple combinatorial formula

For those interested in details: define $\det V = \sum q_i$ weights [$\det V$ tells us about the canonical bundle]. choose λ_w 1-parameter subgroup normal to the wall, oriented s.t.

$\kappa := \det V \cdot \lambda_w > 0$. κ times

λ_w will point into the 'bigger' chamber X.

Toric CY HMS

Take $(\mathbb{C}^*)^r \curvearrowright \mathbb{C}^n$ given by Q

Def A toric GIT problem is Calabi-Yau if the sum of the weights is 0

$$\sum_{i=1}^{n} q_i = 0$$

Wall Crossing implies all GIT quotients are derived equivalent via wall crossing

$$D^b(X) \cong D^b(X')$$

Remark There are countably-infinite **equivalences** $D^b(X) \cong D^b(X')$ through wall crossing

- This implies there are countably-infinitely-many autoequivalences of $D^b(X)$
- [DHL, Shipman] These autoequivalences come from spherical functors $F : D^b(Z) \to D^b(X)$

A note on spherical functors:

They are generalisations of spherical objects $\mathcal{E} \in D^b(X)$ which satisfy

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{E}[i]) : \begin{cases} k & \text{if } i = 0, \dim X \\ 0 & \text{otherwise} \end{cases}$$

where k is the base field. The twist

$$T_{\mathcal{E}} \mathcal{F} := C(\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F})$$

is an autoequivalence.

A spherical functor $F : \mathcal{A} \to \mathcal{B}$ (triangulated cats) satisfy analogous properties, and the twist

$$T_F - := C(FR \to 1_B)$$

where $R : B \to A$ is the right adjoint & $FR \to 1_B$ is the counit of the adjunction

Mirror Symmetry

We have a mirror LG model to the CY GIT toric quotients

$$((\mathbb{C}^*)^{n-r}, W_a : (\mathbb{C}^*)^{n-r} \to \mathbb{C})$$

where W_a is a Laurent polynomial given by the cokernel of Q.

We can study the generic coefficients of W: $a \in (\mathbb{C}^*)^r \setminus \Delta_W$, where Δ_W is the discriminant locus [GKZ]

Def Δ = coefficients of W such that $\exists x \in (\mathbb{C}^*)^{n-r}$ such that $W(x) = dW(x) = 0$

Def Forget Iliopolous Parameter Space (FIPS) $:= (\mathbb{C}^*)^r \setminus \Delta_W$

Symplectic Aside

Def A symplectic manifold (X,ω) is a $2n$-dimensional real manifold X with a non-degenerate 2-form $\omega \in \Omega^2(X)$ (ω called symplectic form)

e.g.: 1) $\mathbb{R}^2_{x_1 x_2}$ $\omega = dx_1 \wedge dx_2$

2) $\mathbb{C}^*$, $\omega = dr \wedge d\theta$

Def A Lagrangian submanifold $L \subseteq X$ a half dimensional submanifold such that $\omega|_L = 0$

e.g. $\mathbb{C}^*$

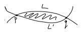
A first, very rough definition (X,ω) symplectic

$\mathrm{Ob}(\mathrm{Fuk}\, X) :=$ Lagrangian submanifolds (with some extra data)

$\mathrm{Hom}_{\mathrm{Fuk}(X)}(L, L') := CF^\bullet(L, L')$ Floer complex

$= \bigoplus_{p \in L \cap L'} \mathbb{C}p$, differential counts holomorphic discs bounded by L & L'

$dp = \sum(\#\text{ discs between } p \,\&\, q)\, q$



This is not $\mathcal{M}_{cx}$, but there is a map FIPS $\to \mathcal{M}_{cx}$

Toric Brane Monodromy

π_1(FIPS) acts on the derived categories of the toric CY GIT quotients

Atiyah Flop

$\mathbb{C}^* \curvearrowright \mathbb{C}^4 \quad Q = (1 \ 1 \ -1 \ -1)$

i.e. $\lambda \cdot (x_1, x_2, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2)$

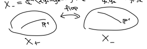
$L \cong \mathbb{Z}$

$\theta > 0 \quad Z_+ = \{x_1 = x_2 = 0\}$

$X_+ = \mathbb{C}^4 \setminus \{x = 0\} / \mathbb{C}^* = \mathcal{O}(-1)^{\oplus 2}_{y_1, y_2} \ \mathbb{P}^1_{x_1 : x_2}$

$\theta < 0 \quad Z_- = \{y_1 = y_2 = 0\}$

$X_- = \mathbb{C}^4 \setminus \{y_1 = y_2 = 0\} / \mathbb{C}^* = \mathcal{O}(-1)^{\oplus 2}_{y_1, y_2} \ \mathbb{P}^1_{y_1 : y_2}$



We have SES

$$0 \to \mathbb{Z} \xrightarrow{Q} \mathbb{Z}^4 \xrightarrow{A^T} \mathbb{Z}^3 \to 0$$

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \\ \begin{array}{ccc} X & Y & Z \end{array} & \end{array} \qquad W_{a,b,c,d}(X, Y, Z) = aXY + bX + cXYZ + dXZ^{-1}$$

The mirror LG model is of the form

$$((\mathbb{C}^*)^3, W_{a,b,c,d} : (\mathbb{C}^*)^3 \to \mathbb{C})$$

where a priori $(a, b, c, d) \in (\mathbb{C}^*)^4$.

Can use A to scale b, c, d to 1,

so FIPS $= \mathbb{C}^*_a \setminus \Delta$

Easy computation gives $\Delta = \{a = 1\}$

FIPS =

π_1(FIPS) $= F(\alpha, \beta)$

$\alpha \leftrightarrow \otimes \mathcal{O}(1)$

$\beta \leftrightarrow T_{\mathcal{O}_{\mathbb{P}^1}}$