Talk

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Derived symmetries induced by crepant contractions

§ Motivation

- Loads of applications

Question: Can we describe the $\mathrm{Auteq}\, D^b(\mathrm{coh}\, X)$?

Theorem (Bondal, Orlov)
X sm projective variety with an ample or antiample canonical bundle then $\mathrm{Auteq}\, D^b(\mathrm{coh}\, X)$ is generated by

(1) $f_* : D^b(\mathrm{coh}\, X) \xrightarrow{\sim} D^b(\mathrm{coh}\, X)$,
f an automorphism.

(2) $(-)\otimes \mathcal{L} : D^b(\mathrm{coh}\, X) \xrightarrow{\sim} D^b(\mathrm{coh}\, X)$
$\mathcal{L}$ a line bundle on X

(3) $[n] : D^b(\mathrm{coh}\, X) \xrightarrow{\sim} D^b(\mathrm{coh}\, X)$
$n \in \mathbb{N}$

$$\Rightarrow \quad \mathrm{Auteq}\, D^b(\mathrm{coh}\, X) \cong \underbrace{\mathbb{Z}}_{\text{Shifts}} \times \left(\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X) \right)$$

$$\phi^*(\mathcal{L} \otimes -) = \phi^*(\mathcal{L}) \otimes \phi^*(-)$$

$\rightsquigarrow$ Point: If the canonical bundle is trivial, $\mathrm{Auteq}\, D^b(\mathrm{coh}\, X)$ is very complicated

"relative" version of this $\rightarrow$ crepant contractions

Philosophy: $\boxed{\text{Birational morphisms w/ trivial}}$ relative canonical bundle induce derived symmetries

§ Crepant contraction

Set up: X, Y noetherian normal varieties over a field k

- Assume Y is quasi-projective
  Y is "Gorenstein"
  $\mathcal{O}_{Y,y}$ finite injective dimension over themselves.
- X, Y are 3D.

Definition
i) A contraction $f : X \to Y$ is a birational morphism satisfying
(a) Projective morphism
(b) $Rf_* \mathcal{O}_X = \mathcal{O}_Y$

ii) A contraction is crepant if
$$f^* \omega_Y = \omega_X$$

Examples, consider $\mathbb{C}^*$ acting on $\mathbb{C}[x_1, x_2, x_3, x_4]$ with weights $(a, b, -s, -t)$
$\lambda \in \mathbb{C}^*$, $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$
$$\lambda * f(x_1, x_2, x_3, x_4) = f(\lambda^a x_1, \lambda^b x_2, \lambda^{-s} \ldots, \lambda^{-t} \ldots)$$

$\lambda \in \mathbb{C}^*$, $f \in$ ...

$\lambda * f(x_1, x_2, x_3, x_4) = f(\lambda^a x_1, \lambda^b x_2, \lambda^{-s} x_3, \lambda^{-t} x_4)$

$S_0 = \mathbb{C}[x_1, \dots, x_4]^{\mathbb{C}^*}$

S_0-modules

$S_i = \{ f \in \mathbb{C}[x_1, \dots x_4] \mid \lambda * f = \lambda^i f \}$

if $a + b - s - t = 0$ then we can construct

(1) $\operatorname{Proj} \underbrace{\bigoplus_{i \geq 0} S_i}_{S_+} \longrightarrow \operatorname{Spec} S_0$

(2) $\operatorname{Proj} \underbrace{\bigoplus_{i \leq 0} S_i}_{S_-} \longrightarrow \operatorname{Spec} S_0$

§ Noncommutative geometry, morphisms with fibre dimension at most 1

<u>Definition</u> (i) A <u>partial tilting bundle</u> on X is a vector bundle $\mathcal{V}$ such that

$\operatorname{Ext}^i_X(\mathcal{V}, \mathcal{V}) = 0 \quad \forall\, i \neq 0$

(ii) A partial tilting bundle $\mathcal{V}$ is a <u>tilting bundle</u> if $\mathcal{V}$ generates $D^b(\operatorname{coh} X)$.

$\mathcal{F} \in D^b(\operatorname{coh} X)$

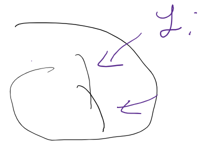
$\operatorname{RHom}_X(\mathcal{V}, \mathcal{F}) = 0 \Rightarrow \mathcal{F} = 0$

<u>Theorem</u> [Van den Bergh]

(i) Let $f: X \to \operatorname{Spec} R$ be a crepant contraction with fiber dimension ≤ 1. Then X admits a tilting bundle $\mathcal{V}$,

$\mathcal{V} = \mathcal{O}_X \oplus \mathcal{V}_0$

$0 \to \mathcal{O}_X^r \to \mathcal{V}_0 \to \mathcal{L} \to 0$



(ii) A tilting bundle induces a derived equivalence

$\operatorname{RHom}_X(\mathcal{V}, -) : D^b(\operatorname{coh} X) \longrightarrow D^b(\operatorname{mod} \Lambda^{op})$

$\Lambda = \operatorname{End}_X(\mathcal{V})$

$\Lambda = \mathrm{End}_X(\mathcal{V})$

(iii) $\Lambda = \mathrm{End}_X(\mathcal{V}) \cong \mathrm{End}_R(f_*\mathcal{V})$

$\cong \mathrm{End}_R(R \oplus f_*\mathcal{V}_0)$

§ Contraction Algebra

$f: X \to \operatorname{Spec} R$ as before.

derived equivalent

$\Lambda = \mathrm{End}_X(\mathcal{V}) \cong \mathrm{End}_R(f_*\mathcal{V})$

Definition The contraction algebra is the quotient $\Lambda_{con} := \Lambda / I$

$I = \langle$ maps $f_*\mathcal{V} \to f_*\mathcal{V}$ which factor through proj $R \rangle$

R Gorenstein

CM R

$\Lambda = \underset{R}{\mathrm{End}}(f_*\mathcal{V})$

$\Lambda_{con} = \mathrm{End}_{\underline{CM} R}(f_*\mathcal{V})$

$I \subset \mathrm{End}_X(\mathcal{V})$

$s \in \mathrm{End}(\mathcal{V})$

$I = \{ s \mid s_x$ factors through proj $\mathcal{O}_{X,x}$ module $\}$

§ Derived symmetries

$f: X \to \operatorname{Spec} R$ as before.

$\rightsquigarrow X$ has a tilting bundle $\mathcal{V}$ which induces a derived equivalence

$\phi := \mathrm{RHom}_X(\mathcal{V}, -): D^b(\mathrm{coh}\, X) \xrightarrow{\sim} D^b(\mathrm{mod}\, \Lambda^{op})$

where $\Lambda = \mathrm{End}_R(f_*\mathcal{V})$

$I \trianglelefteq \Lambda$

$\hookrightarrow$ ideal of maps $f_*\mathcal{V} \to f_*\mathcal{V}$ which factor through proj R.

Definition The noncommutative twist is

$\Upsilon := \mathrm{RHom}_\Lambda(I, -): D^b(\mathrm{mod}\, \Lambda^{op}) \to D^b(\mathrm{mod}\, \Lambda^{op})$

$\rightsquigarrow$ when is is Υ an equivalence?

$$\begin{tikzcd} D^b(\mathrm{coh}\, X) \arrow[r, "\phi", "\sim"'] & D^b(\mathrm{mod}\, \Lambda^{op}) \arrow[d, "\Upsilon", "?"'] \\ & D^b(\mathrm{mod}\, \Lambda^{op}) \arrow[ul, "\phi^{-1}", "\sim"'] \end{tikzcd}$$

would give ... on X

$\xrightarrow[\sim]{\Phi^{-1}} D^b(\text{mod } \Lambda^{op}$

would give a derived symmetry of X

In our running examples:

(1) Υ is an equivalence

(2) + (3) Υ is not an equivalence

What goes wrong in (2) and (3)?

<u>Theorem</u> For Υ to be an equivalence we require the contraction

$f : X \longrightarrow \operatorname{Spec} R$

to satisfy 2 properties

(1) $\operatorname{pdim}_\Lambda \Lambda_{con} < \infty$ $\rightsquigarrow$ "hidden smoothness"

(2) Λ_{con} to be self-injective (i.e. Λ_{con} to be injective as a module over itself) $\rightsquigarrow$ "spherical criterion"

If (1) + (2) are satisfied, then Υ is a spherical twist around the functor

$- \otimes_\Lambda \Lambda_{con} : D^b(\text{mod } \Lambda_{con}) \longrightarrow D^b(\text{mod } \Lambda)$

Ex 1

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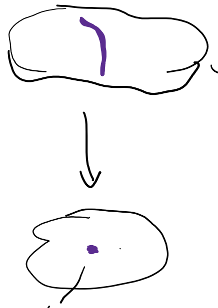
(1) weights $(1, 1, -1, -1)$ (Atiyah flop)

$$\begin{array}{c} \operatorname{Proj} S_+ \\ \Big\downarrow f \quad \text{blow up of } Y \text{ along the ideal } (a, c) \\ Y = \operatorname{Spec} S_0 = \operatorname{Spec} \dfrac{\mathbb{C}[a,b,c,d]}{(ab-cd)} \end{array}$$

$X \wr$  $\to$ smooth $\rightsquigarrow$ Λ has finite global dimension

$Y =$ isolated sing.

$\rightsquigarrow$ VdB tells us that $X = \operatorname{Proj} S_+$ admits a tilting bundle

$$\mathcal{V} = \mathcal{O}_X \oplus \mathcal{V}_0$$

$$f_* \mathcal{V} \cong S_0 \oplus S_1$$

$$\Lambda = \operatorname{End}_R(S_0 \oplus S_1)$$

$$\Lambda^{op} = \operatorname{End}_R(S_0 \oplus \operatorname{Hom}_{S_0}(S_1, S_0))$$

$$= \operatorname{End}_R(S_0 \oplus S_{-1})$$

Λ^{op} looks like:

$$\begin{tikzcd} S_0 \arrow[r, bend left=50, "c"] \arrow[r, bend left=25, "a"] & S_{-1} \arrow[l, bend left=25, "b/c"] \arrow[l, bend left=50, "d/a"] \end{tikzcd}$$

Derived equiv $\cong$ $\operatorname{Proj} S_+$

$\Big\}$ delete s_0

Λ_{con} : $\underset{s_{-1}}{\circ}$ $\rightsquigarrow$ $\Lambda_{con} \cong \mathbb{C}$ as algebras.

(1) $\operatorname{pdim}_{\Lambda} \Lambda_{con} < \infty$ since Λ is smooth ✓

(2) Λ_{con} is self-injective ✓

$\Rightarrow \Psi = \mathbf{R}\mathrm{Hom}(I, -)$ is an equivalence ✓

Ex 2

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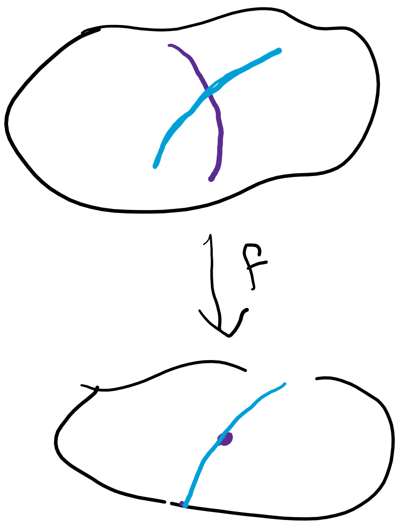
(2) $\underline{w} = (3, 1, -2, -2)$

$$\begin{array}{c} \mathrm{Proj}\ S_t \\ \downarrow \\ Y = \operatorname{Spec} \dfrac{\mathbb{C}[x_1, \dots x_q]}{J} \end{array}$$

⇝ blow up along the ideal (x_p, x_q)



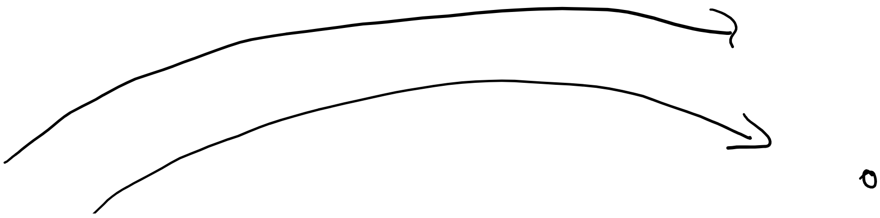
$= X$ ⇝ no longer smooth

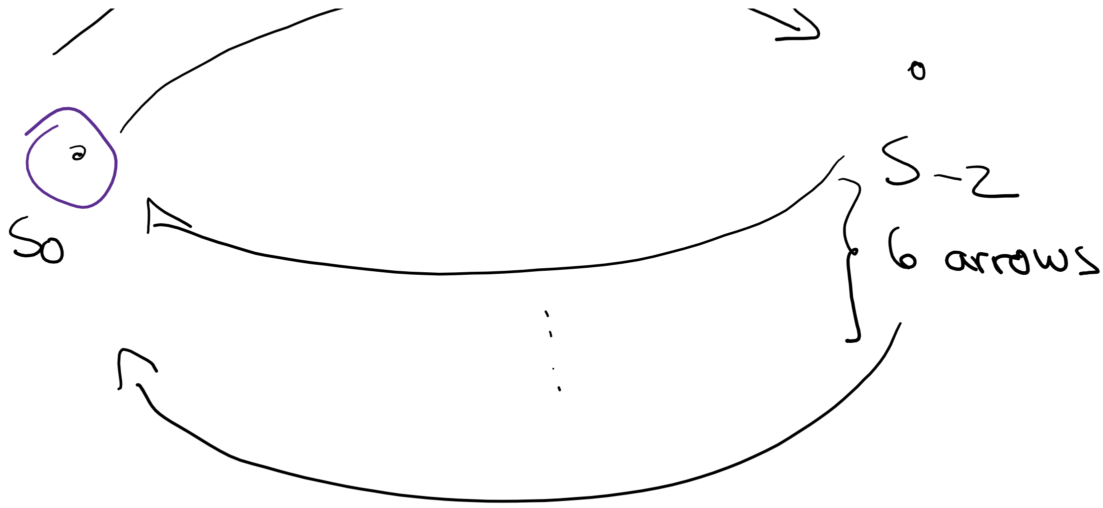
Singularities

$$f_* \mathcal{V} = \mathrm{End}_{S_0}(S_0 \oplus S_2)$$

⇝

Λ^{op} looks like





Λ_{con}:

$\overset{\circ}{S_{-2}}$ $\quad \rightsquigarrow \quad \underbrace{\Lambda_{con} \cong \mathbb{C}}$

(1) $\operatorname{pdim} \Lambda_{con} = \infty$ ✗

(2) Λ_{con} is self injective.

Ex 3

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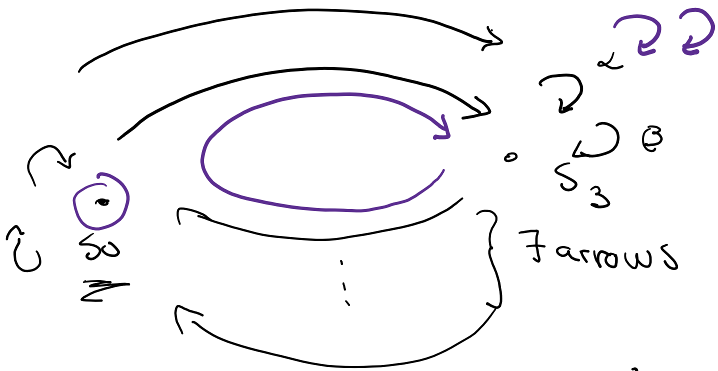
(3) $(3, 1, -2, -2)$

$\mathrm{Proj}\ S_- \leftarrow X$ is no longer smooth

$\downarrow$

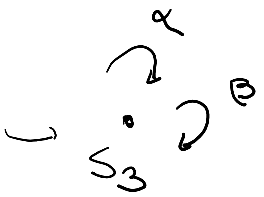
$\mathrm{Spec}\ \dfrac{\mathbb{C}[x_1, \dots, x_9]}{J}$

Λ^{op} looks like:



+ relations: α^2, β^2 and $\alpha\beta$ then they factor through S_0

Λ_{con}:



$\rightsquigarrow$ $\alpha^2 = 0$, $\beta^2 = 0$, $\alpha\beta = 0$

+ relations $\rightsquigarrow$

$\underbrace{\Lambda_{con} \cong \mathbb{C}[x, y] / (x^2, y^2, xy)}$

↳ not self-injective

(2) Λ_{con} is not self-injective ✗

(1)