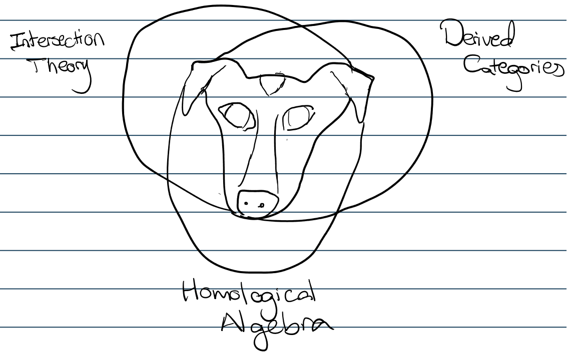
DOGS 28/05

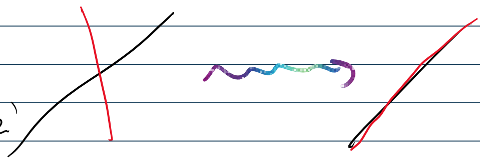


What the hell is derived intersection theory?

I don't know :(

Intersection Thy

Want sensible numerics in enumeration problems following a principle of 'continuity'



Solⁿ Consider 'cycles' upto 'rational equivalence'

'Weil divisors' 'linear equivalence'

Define intersection product of divisors as follows

$$\begin{tikzcd} & V \arrow[d, hook, "i"] \\ D \arrow[r, hook] & X \end{tikzcd}$$

$D \rightsquigarrow$ line bundle on X, $\mathcal{O}(D)$

$i^*\mathcal{O}(D)$ line bundle on $V \rightsquigarrow$ divisor in V
'codimension 1 cycle'

Weird to me that $D \cdot D$ is somehow codimension 1 in the divisor

Can generalise to higher codim

Excess Intersection Formula

$$\begin{tikzcd} V \cap W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & V \arrow[d, hook] \\ W \arrow[r, hook] & X \end{tikzcd}$$

Excess normal directions

$0 \to \mathcal{N}_{V\cap W/V} \hookrightarrow (\mathcal{N}_{W/X})|_{V\cap W} \to E^\vee \to 0$

W, V are local complete intersections in X
&
$W \cap V$ is a lci in V

$V \cdot W = c_{\underbrace{d-d'}_{\text{rank}(E^\vee)}}(E^\vee) \cap [V \cap W]$

$d = \text{codim}(W, X)$, $d' = \text{codim}(V \cap W, V)$

Derived

In any category an intersection of subobjects is given by a pullback

$$\begin{tikzcd} V \cap W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & V \arrow[d, hook] \\ {} & {} \end{tikzcd}$$

pullback
$$\begin{tikzcd} V\cap W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & V \arrow[d, hook] \\ W \arrow[r, hook] & X \end{tikzcd}$$

Topology, $V\cap W$ = set-theoretic intersection

Scheme theory $(V,\mathcal{O}_V)\cap(W,\mathcal{O}_W) = (V\cap W, \mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_W)$

Intersection structure.

Derived scheme theory $(V,\mathcal{O}_V)\cap(W,\mathcal{O}_W) = (V\cap W, \underbrace{\mathcal{O}_V \otimes^L_{\mathcal{O}_X} \mathcal{O}_W}_{\in D(X)})$

$$\mathcal{O}_V\otimes^L \mathcal{O}_W \cong P_V^\bullet \otimes \mathcal{O}_W \cong \mathcal{O}_V\otimes P_W^\bullet \cong P_V^\bullet\otimes P_W^\bullet$$

Here $P_V^\bullet, P_W^\bullet$ are flat resolutions of $\mathcal{O}_V, \mathcal{O}_W$

In the case of $D\subset X$ an effective Cartier divisor

Scheme theory

$(D,\mathcal{O}_D)\cap(D,\mathcal{O}_D)\cong(D,\mathcal{O}_D\otimes\mathcal{O}_D)$

$\cong(D,\mathcal{O}_D)$

Derived

$(D,\mathcal{O}_D\otimes^L\mathcal{O}_D)$

$\cong\Big(D,\{\underbrace{\mathcal{O}_D(-D)}_{\substack{\text{measures}\\ \text{excess}\\ C_{D/X}}}\xrightarrow{0}\underbrace{\mathcal{O}_D}_{\text{classical}}\}\Big)$

Comparison

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_V,\mathcal{O}_W) = H^{-q}(\mathcal{O}_V\otimes^L\mathcal{O}_W)$$

Th$^{\underline{m}}$ (Serre)

X regular variety, $V,W\subseteq X$ complementary dimension s.t. $\dim(X\cap V)=0$, Then

$$\underbrace{i(P,V,W;X)}_{\text{Can define via Fulton}} = \sum_{i=0}^{\infty}(-1)^i\,\mathrm{length}_{\mathcal{O}_{X,P}}\mathrm{Tor}_i^{\mathcal{O}_{X,P}}(\mathcal{O}_{V,P},\mathcal{O}_{W,P})$$

More generally:

$$\exists \text{ GRR iso } \tau: K_0(X)_{\mathbb{Q}} \xrightarrow{\sim} \overset{\text{cycles/rat.eq.}}{\overset{\|}{A_*(X)_{\mathbb{Q}}}}$$

For any $V,W\subseteq X$, define $\mathrm{Tor}^X(V,W)=\sum(-1)^i[\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V,\mathcal{O}_W)]\in K_0(X)$

Then $\tau(\mathrm{Tor}^X(V,W)) = V\cdot W +$ lower dim terms.

Excess Intersection Formula [Scala '15]

Th$^{\underline{m}}$

Excess Intersection Formula [Scala '15]

Th$^{\underline{m}}$ Let $Y_1, \dots, Y_n \subseteq X$ be lci subvarieties of X nonsing. $/k = \bar{k}$ char $k = 0$.

Assume $\bigcap Y_i =: Z$ is lci in X. Define E by

$$0 \to E \to \left(\bigoplus C_{Y_i/X}\right)\Big|_Z \to C_{Z/X} \to 0$$

Excess conormal directions

Then $\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) \cong \Lambda^q E$.

$$\mathrm{Tor}_q(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_n}) \cong H^{-q}(\mathcal{O}_{Y_1} \otimes^L \dots \otimes^L \mathcal{O}_{Y_n}) \cong H^{-q}(P^\bullet_{Y_1} \otimes \dots \otimes P^\bullet_{Y_n})$$

Special case is self-intersection formula

$$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \cong \Lambda^q C_{Y/X}$$

$\mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)$ is only really defined up to isomorphism

Need to pick flat models $P_V^\bullet \to \mathcal{O}_V$, $P_W^\bullet \to \mathcal{O}_W$ then $\mathrm{Tor}_q(\mathcal{O}_V, \mathcal{O}_W) := H^{-q}(P_V^\bullet \otimes P_W^\bullet)$

Picking different flat models gives rise to an isomorphism

$$H^{-q}(P_V^\bullet \otimes P_W^\bullet) \to H^{-q}(Q_V^\bullet \otimes Q_W^\bullet)$$

Local complete intersections = Good because they locally have bdd free resolutions

$\mathrm{Spec}(A)$, I_Y defined by regular sequence $f_1, \dots, f_{r = \mathrm{codim}(Y,X)}$

$K^\bullet(f_1, \dots, f_r)$ defined by $K^{-q}(f_1, \dots, f_r) = \Lambda^q A^r \cong A^{\binom{r}{q}}$

$$d(a_1 \wedge \dots \wedge a_q) = \sum (-1)^i f_i \, a_1 \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge a_q$$

$K^\bullet(f_1)$ $\quad A \xrightarrow{f_1} A$

$K^\bullet(f_1, f_2) =$

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] & \\
A \arrow[ur, "f_2"] \arrow[dr, "-f_1"'] & \oplus & A \\
 & A \arrow[ur, "f_2"'] &
\end{tikzcd}$$

$K^\bullet(f_1, f_2, f_3)$

$$\begin{tikzcd}
 & A \arrow[r, "f_2"] \arrow[dr, "-f_1"] & A \arrow[dr, "f_1"] & \\
A \arrow[ur, "f_3"] \arrow[r, "-f_2"] \arrow[dr, "f_1"'] & A \arrow[ur, "f_3"] \arrow[dr, "-f_1"] & A \arrow[r, "f_2"] & A \\
 & A \arrow[ur, "f_3"] \arrow[r, "-f_2"'] & A \arrow[ur, "f_3"'] &
\end{tikzcd}$$

Sequence $f_1, \dots, f_r$ regular $\iff K^\bullet(f_1, \dots, f_r) \to A/(f_1, \dots, f_r)$ a resolution.

Computation

V, W lci $\subseteq X$ nonsing$/k = \bar{k}$

V, W lci $\subseteq X$ holding$/ k = \bar{k}$

Pick global flat res $P_V^\cdot \to \mathcal{O}_V$, $P_W^\cdot \to \mathcal{O}_W$.

Local Koszul models $K^\cdot(f_1, \dots, f_r) \to \mathcal{O}_V|_U$

$K^\cdot(g_1, \dots, g_s) \to \mathcal{O}_W|_U$

Freeness implies unique lifts (up to homotopy)

$$\begin{tikzcd} & P_V^\cdot|_U \arrow[d] & P_W^\cdot|_U \arrow[d] & \\ K^\cdot(f_1,\dots,f_r) \arrow[ur, dashed] \arrow[r] & \mathcal{O}_V|_U & \mathcal{O}_W|_U & K^\cdot(g_1,\dots,g_s) \arrow[l] \arrow[ul, dashed] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

induces a quasi-isomorphism

$$K^\cdot(f_1,\dots,f_r) \otimes K^\cdot(g_1,\dots,g_s) \to P_V^\cdot|_U \otimes P_W^\cdot|_U$$ because of q-flatness

Hence isomorphisms $\cong$

$$\varphi : H^{-q}(K^\cdot(f_1,\dots,f_r,g_1,\dots,g_s)) \to \mathrm{Tor}_q(\mathcal{O}_V,\mathcal{O}_W)|_U$$

What if we pick different generators?

We get a 'change of basis' map $\chi : K^\cdot(f_1,\dots,f_r,g_1,\dots,g_s) \to K^\cdot(f_1',\dots,f_r',g_1',\dots,g_s')$

s.t.

$$\begin{tikzcd} & \mathrm{Tor}_q(\mathcal{O}_V,\mathcal{O}_W)|_U & \\ H^{-q}(K^\cdot(\underline{f},\underline{g})) \arrow[ur, "\varphi", "\sim"'] \arrow[rr, "H^{-q}\chi", "\sim"'] \arrow[rr, phantom, bend left=20, "\circlearrowleft"] & & H^{-q}(K^\cdot(\underline{f}',\underline{g}')) \arrow[ul, "\varphi'"', "\sim"] \end{tikzcd}$$

$$\begin{tikzcd} H^{-q}(K^\cdot(\underline{f},\underline{g})) \arrow[dr, "\sim"] \arrow[rr, phantom, "\circlearrowleft"] & & H^{-q}(K^\cdot(\underline{f}',\underline{g}')) \arrow[dl, "\sim"'] \\ & \mathcal{F}|_U & \end{tikzcd}$$

$\Rightarrow \mathrm{Tor}_q(\mathcal{O}_V,\mathcal{O}_W) \cong \mathcal{F}$

Sounds easy and boring

Boring ✓

Self-intersection

Isomorphisms $H^{-q}(K^\cdot(f_1,\dots,f_r) \otimes A/(f_1,\dots,f_r)) \cong K^{-q}(f_1,\dots,f_r) \otimes A/(f_1,\dots,f_r)$

$\cong \Lambda^q C_{Y/X}|_U$

+ gluing data

This is not hard

Same philosophy is used to prove EIF but much harder to show gluing because the local isomorphisms are more involved.

Same philosophy is used to prove E+T but much harder to show gluing because the local isomorphisms are more involved.

Reduce to the diagonal

$$\mathcal{O}_{Y_1} \otimes^L \cdots \otimes \mathcal{O}_{Y_n} \cong \Delta^* \mathcal{O}_{Y_1 \times \cdots Y_n}$$

$$\begin{tikzcd} \bigcap Y_i \arrow[r, "k"] \arrow[d, "w"'] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "\Delta"] \\ Y_1 \times \cdots Y_n \arrow[r, hook, "i_1 \times \cdots \times i_n"'] & X^{\times n} \end{tikzcd}$$

$$\Delta^* \mathcal{O}_{Y_1 \times \cdots \times Y_n} \xrightarrow{\eta_k} k_* k^* \Delta^* \mathcal{O}_{Y_1 \times \cdots \times Y_n}$$
$$\cong k_* \underbrace{w^* (i_1 \times \cdots \times i_n)^* (i_1 \times \cdots \times i_n)_* \mathcal{O}_{Y_1 \times \cdots \times Y_n}}_{\text{self-intersection of the lci } Y_1 \times \cdots \times Y_n \hookrightarrow X \times \cdots \times X}$$

$$i_* \mathcal{O}_Y \otimes^L i_* \mathcal{O}_Y \cong L i^* i_* \mathcal{O}_Y$$

Global diagram

$$\begin{tikzcd} \mathrm{Tor}_q(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_n}) \arrow[r, "H^{-q}\eta_k"] & H^{-q}(k_* k^* \Delta^* \mathcal{O}_{Y_1 \times \cdots \times Y_n}) \arrow[d, "\wr \varphi"] \\ \Lambda^q E \arrow[r, hook] & \Lambda^q(\bigoplus C_{Y_i/X})|_W \end{tikzcd}$$

Just need to check that

1) $H^{-q}\eta_k$ is injective

2) Its image agrees with the image of $\Lambda^q E$

These checks are local $\therefore$ no gluing required