An introduction to Bridgeland stability conditions on threefolds

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See-saw Property

Let $0 \to F \to E \to G \to 0$ be a short exact sequence in $\mathrm{Coh}(C)$. Then, either

$$\mu(F) < \mu(E) < \mu(G);$$
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- A tool to produce Moduli Spaces of $\mathrm{Coh}(C)$
  For instance, $V_r(C)$ is a non-singular quasi-projective variety

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Definition

An Harder-Narasimhan filtration for $E \in \mathrm{Coh}(C)$ is

$$\begin{tikzcd}
0 = E_0 \arrow[r, hook] \arrow[dr, "0"] & E_1 \arrow[r, hook] \arrow[d, two heads] \arrow[dr, "0"] & E_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] \arrow[dr, "0"] & E_n = E \arrow[d, two heads] \\
& A_1 & A_2 & & A_n
\end{tikzcd}$$

with $A_1, \cdots, A_n$ μ-semistable and $\mu(A_1) > \mu(A_2) > \cdots > \mu(A_n)$.

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- otherwise, take $G' \subset G$ the largest subsheaf of G, then

$$\begin{tikzcd}
E_1 \arrow[r, hook] \arrow[dr, hook] & E \arrow[r, two heads] & G \\
 & E_2 \arrow[u, hook] \arrow[r, two heads] & G' \arrow[u, hook]
\end{tikzcd}$$

with $\mu(E/E_2) = \mu(G/G')$ and $\mu(E_1) > \mu(E/E_2)$. $\square$

Higher dimensions

Let X be a smooth projective n-dimensional variety, $H \in \mathrm{Pic}(X)$ an ample line bundle. For $E \in \mathrm{Coh}(X)$, define

$$\mu_H(E) := \frac{H^{n-1} \cdot c_1(E)}{rk(E)}.$$

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Solutions:

- Define polynomial stability conditions (Gieseker)
- Try to enlarge it

General Bridgeland Stability

Let $\mathcal{D}$ be a triangulated category.

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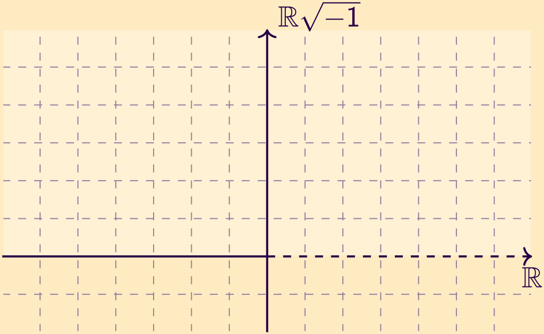
  2. Every $E \in \mathcal{A}$ has an Harder-Narasimhan filtration
  3. (Support Property)

$$C_\sigma := \inf \left\{ \frac{|Z(E)|}{\|\nu(E)\|} \;:\; 0 \neq E \in \mathcal{A} \text{ semistable} \right\} > 0$$

General Bridgeland Stability



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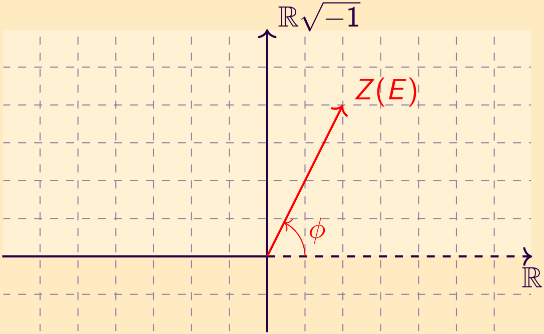

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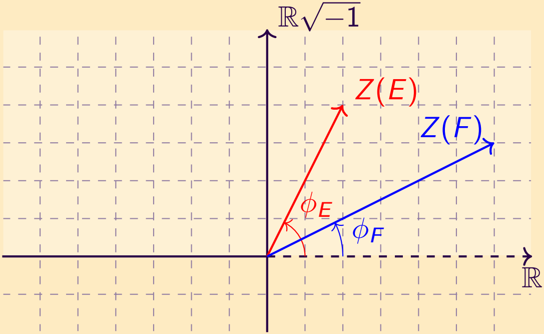

- $Z(E) \in \mathbb{R}_{>0} \cdot e^{\pi\phi\sqrt{-1}}$ with $\phi \in (0, 1]$
- $E \in \mathcal{A}$ is (semi)stable if, for all non-trivial $F \subset E$, $\phi_F < (\leq)\, \phi_E$

Bridgeland's Deformation Theorem

Theorem (Bridgeland's Deformation Theorem)

The natural map

$$\mathcal{Z} : \mathrm{Stab}(X) \to \mathrm{Hom}(\Lambda, \mathbb{C}), \quad \sigma = (\mathcal{A}, Z) \mapsto Z$$

is a local homeomorphism. In particular, $\mathrm{Stab}(X)$ is a complex variety of dimension $\mathrm{rk}(\Lambda)$.

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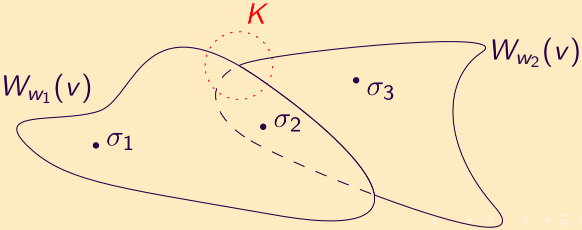
For any fixed class $v \in \Lambda$, $\mathrm{Stab}(X)$ has a natural walls-chambers structure:



How to build a BSC

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$$A(X) = \begin{cases} A^0(X) = \mathbb{Z} \cdot [X] & \ni ch_0 \\ A^1(X) \cong \mathrm{Pic}(X) & \ni ch_1 \\ A^2(X) & \ni 2 \cdot ch_2 \\ A^3(X) & \ni 6 \cdot ch_3 \\ \vdots & \\ A^n(X) \cong \mathbb{Z} \cdot [p] & \text{(for Fano)} \end{cases}$$

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For instance, if X is a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}$, then

$$\Lambda = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \frac{1}{6}\mathbb{Z}$$

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- **Big Problem:** To find $\mathcal{A} \subset \mathrm{D}^b(X)$ and $Z : K(\mathcal{A}) \to \mathbb{C}$ is done case-by-case.

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  3. We need some inequality to verify the Support Property

BSC on Curves

Let X be a smooth projective curve of genus one.

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- $\mathcal{A} = \mathrm{Coh}(X)$ is the heart of the bounded t-structure

$$\mathcal{D}^{\leq 0} = \{X \in \mathrm{D}^b(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i > 0\}$$
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Theorem (Bridgeland)

The space of BSC is isometric to $\mathrm{Stab}(X) \cong \tilde{GL}^{+}(2, \mathbb{R})$, the universal covering space of $GL^{+}(2, \mathbb{R})$.

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This cannot work: let $x \in X$ be a closed point, then

$$\mathrm{ch}_0(k(x)) = \mathrm{ch}_1(k(x)) = 0 \implies Z(k(x)) = 0$$

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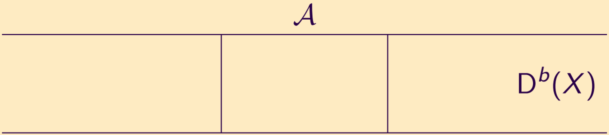

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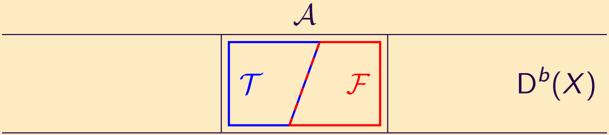

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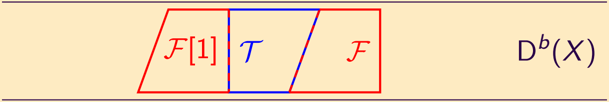

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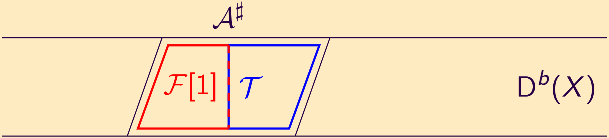

BSC on Surfaces

For $\beta \in \mathbb{R}$, consider the torsion pair

$$\mathcal{T}_\beta := \{E \in \operatorname{Coh} X \mid \forall E \twoheadrightarrow G \neq 0,\ \phi_Z(G) > \beta\};$$

$$\mathcal{F}_\beta := \{E \in \operatorname{Coh} X \mid \forall 0 \neq F \hookrightarrow E,\ \phi_Z(F) \leq \beta\}.$$

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$$\mathcal{H}^0(B) \in \mathcal{T}_\beta,\ \mathcal{H}^{-1}(B) \in \mathcal{F}_\beta, \text{ and } \mathcal{H}^i(B) = 0 \text{ for } i \neq 0, -1.$$

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Take $\alpha \in \mathbb{R}^{+}$, define the group homomorphism

$$Z^{\mathrm{tilt}}_{\alpha,\beta}(B) := -\left(\mathrm{ch}_2^{\beta}(B) - \frac{1}{2}\alpha^2\,\mathrm{ch}_0(B)\right) + \sqrt{-1}\,\mathrm{ch}_1^{\beta}(B)$$

for $B \in \mathcal{B}^{\beta}$.

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Theorem (Bridgeland)

Let X be a K3 surface. Then the pair $\sigma_{\alpha,\beta} = (\mathcal{B}^\beta, Z^{tilt}_{\alpha,\beta})$ is a BSC.

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Lemma (Bogomolov-Gieseker inequality)

Any $B \in \mathcal{B}^\beta$ satisfies

$$Q^{tilt}(B) := \mathrm{ch}_1(B)^2 - 2\,\mathrm{ch}_0(B)\,\mathrm{ch}_2(B) \geq 0.$$

$\square$

BSC on 3-folds

For X a smooth projective 3-fold, we tilt again:
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Take $s \in \mathbb{R}^+$, we define the group homomorphism

$$Z_{\alpha,\beta,s}(A) = -\operatorname{ch}^\beta_3(A) + \left(s + \frac{1}{6}\right)\alpha^2 \operatorname{ch}^\beta_1(A) + \sqrt{-1}\left(\operatorname{ch}^\beta_2(A) - \frac{1}{2}\alpha^2 \operatorname{ch}_0(A)\right)$$

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Theorem (Bayer, Macrì, Toda)

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Conjecture (generalized Gieseker-Bogomolov)

Let X be a Fano threefold, if $B \in \mathcal{B}^{\beta}$ is $Z^{tilt}_{\alpha,\beta}$-semistable, then

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- False for $X = \mathrm{Bl}_p(\mathbb{P}^3)$ (Schmidt, 2017)

Asymptotic stability: the tool

Definition (Jardim, Maciocia, Martinez)

Let $\gamma : [0, \infty) \to \mathbb{H}$ be a path, an object $A \in \mathrm{D}^b(X)$ is asymptotic $Z_{\alpha,\beta,s}$-(semi)stable along γ if, for a fixed $s > 0$, the following conditions hold:

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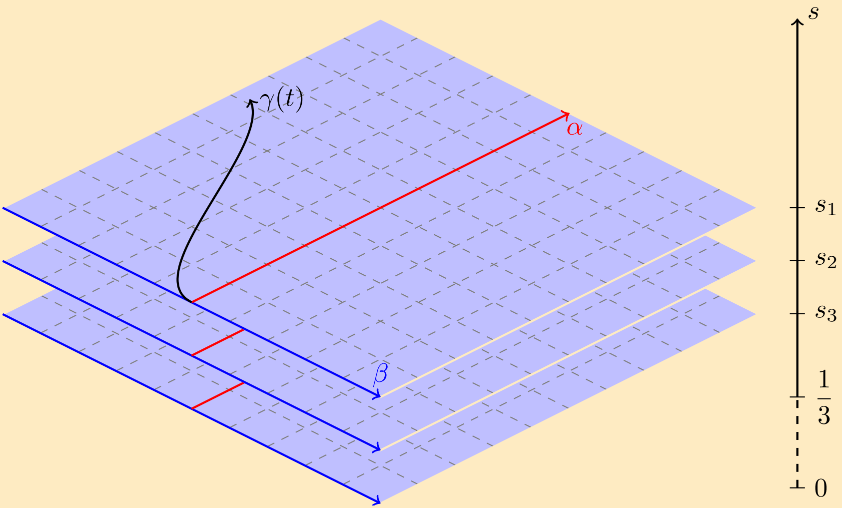
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β
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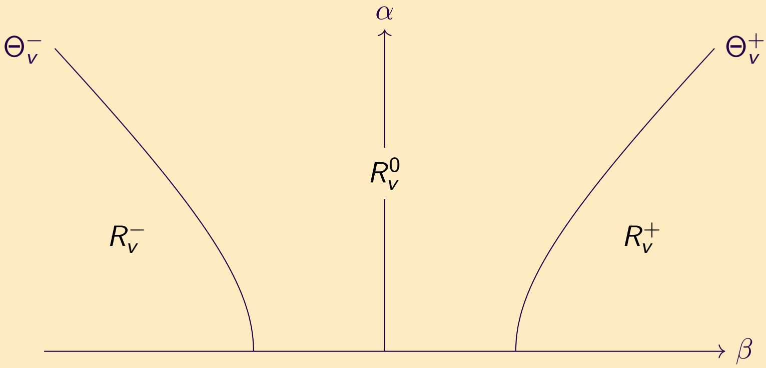

Figure: Numerical wall for $v = (2, 0, -3, 0)$.

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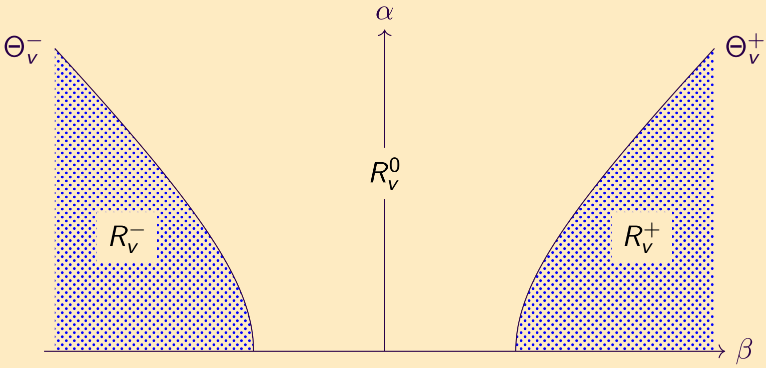

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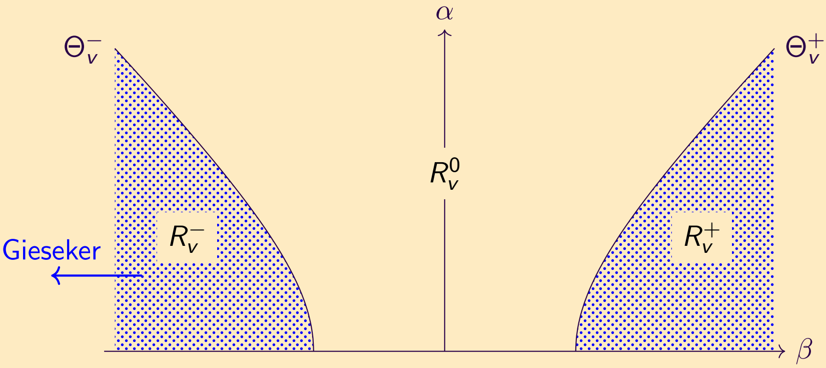

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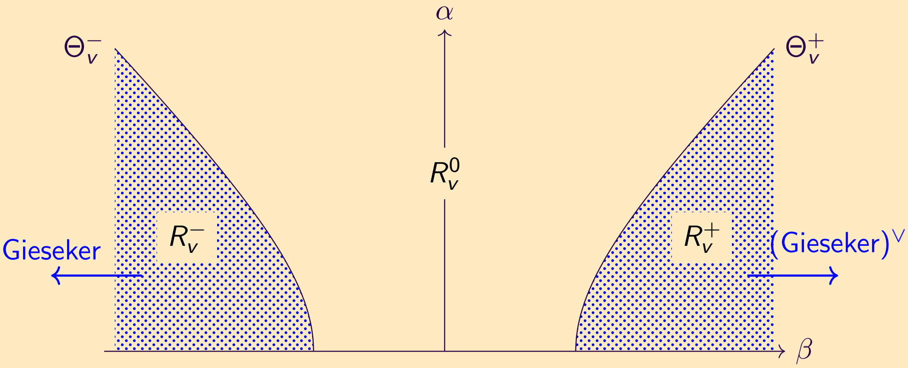

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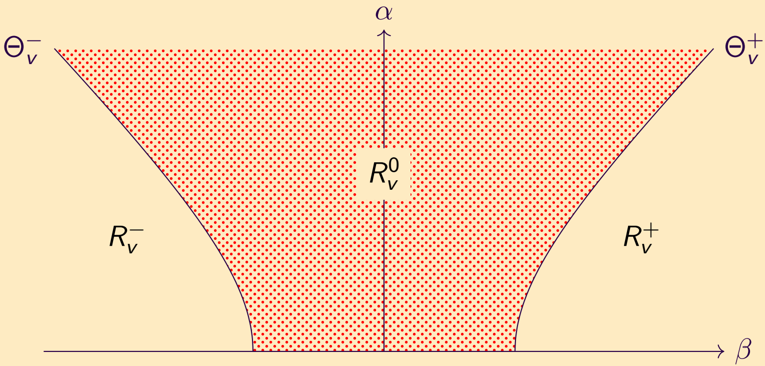

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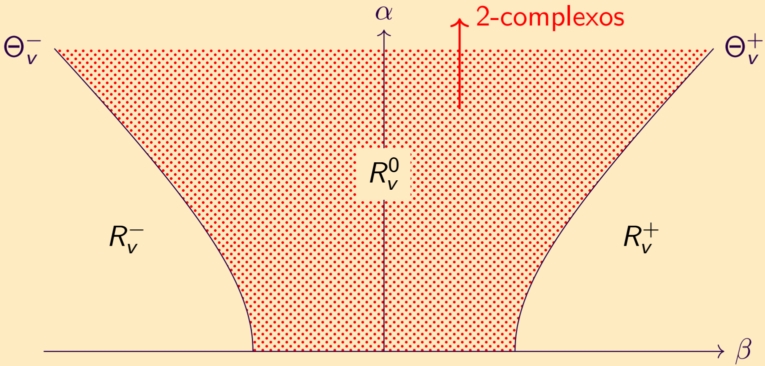

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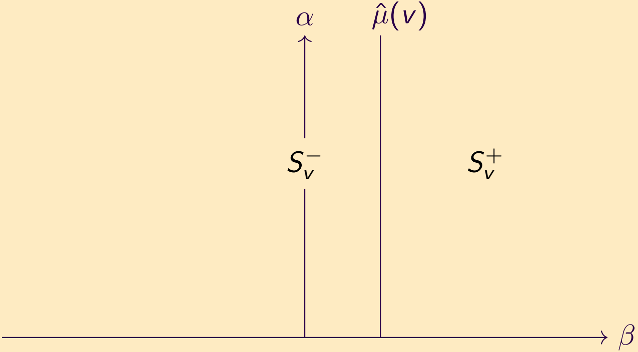

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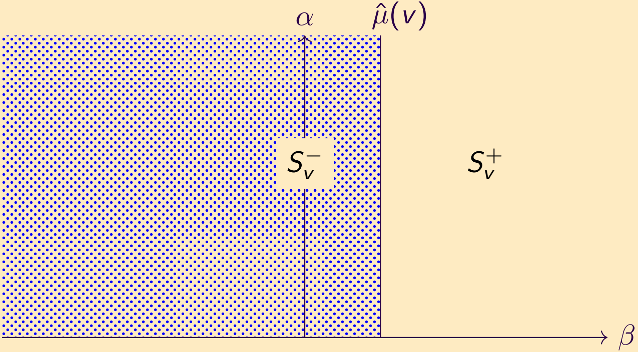

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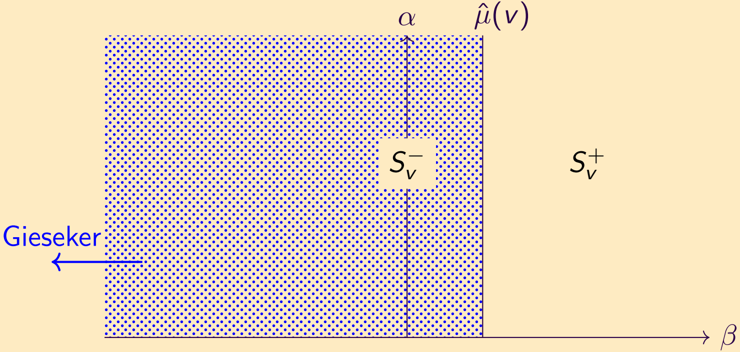

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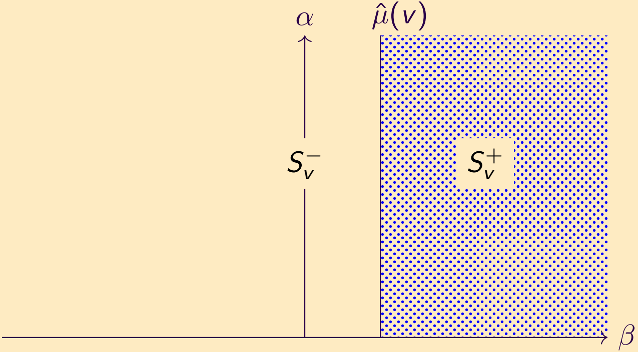

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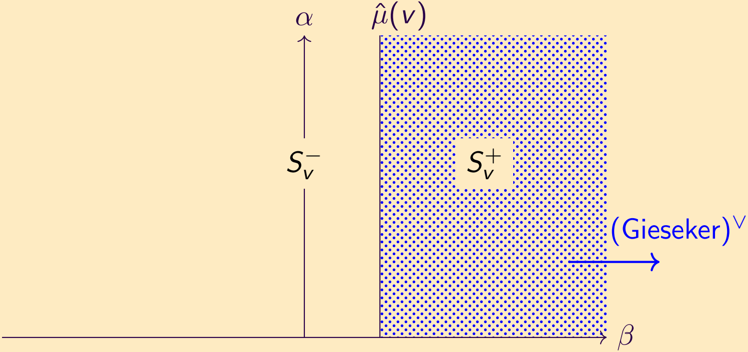

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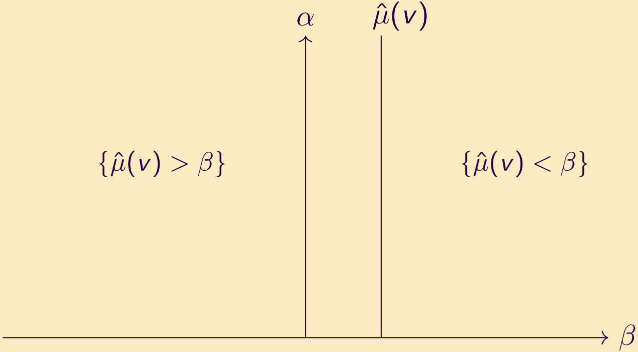

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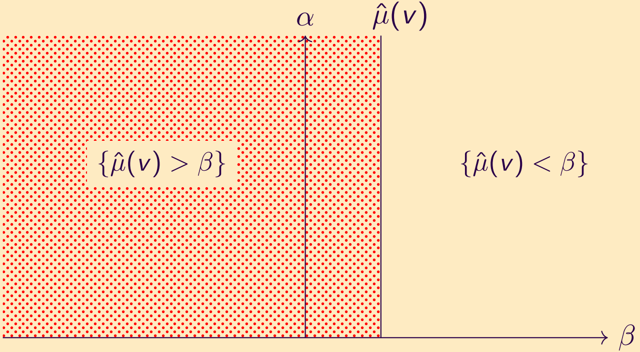

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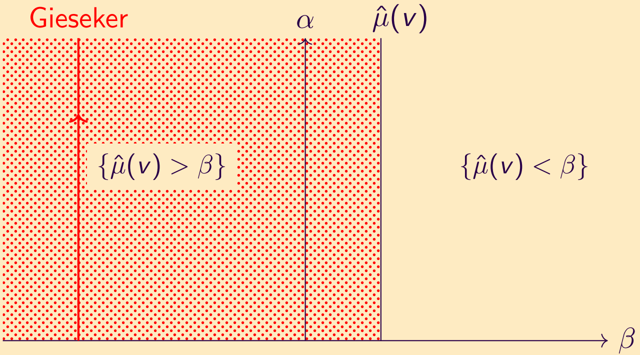

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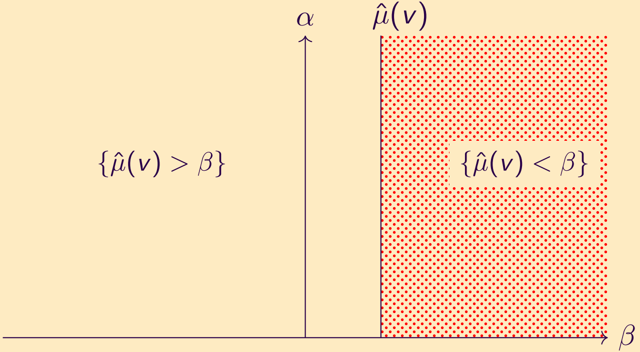

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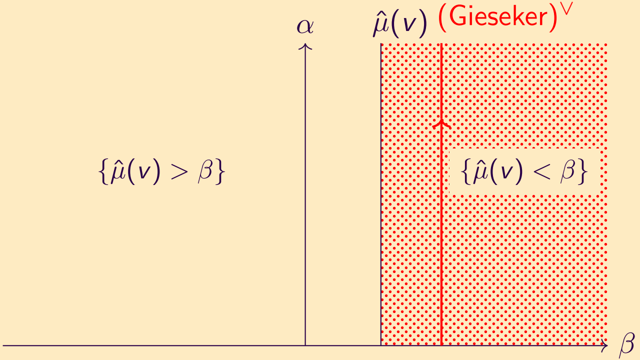

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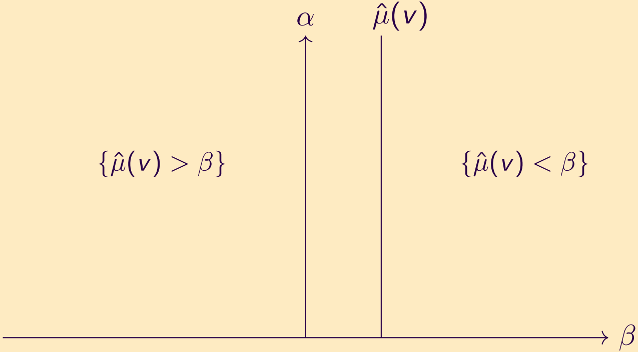

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Fun fact: This result is actually mine! :-)

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- Applications:
  Higher bound for globals sections on curves
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Thank you for watching!

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