(Q): Why is $\mathrm{Stab}\,\mathbb{P}^1 \simeq \mathbb{C}^2$?

$Z_\mu = \deg + i\,\mathrm{rank}$

$\dim X = 1$, $\mathrm{Stab}(X) \supset (\mathrm{Coh}\,X, Z_\mu)\cdot \widetilde{GL_2^+}(\mathbb{R}) = \mathrm{Stab}^{\mathrm{Geo}}(X)$

① $\mathcal{O}_x$ is σ_μ-stable $\Leftrightarrow$ σ_μ "geometric"

$\phi \in (0,1]$, $\mathcal{E} \in \mathrm{Coh}\,X$ μ-semistable $\Leftrightarrow$ $\mathcal{E} \in P(\phi)$, $Z(\mathcal{E}) = re^{i\pi\phi}$

$\phi \in (n, n+1]$, objects in $P(\phi)$: $\mathcal{E}[n]$, $\mathcal{E}$ is μ-semistable

$\widetilde{GL_2^+}(\mathbb{R}) \underset{\text{as } \mathbb{C}\text{-mfds}}{\simeq} \mathbb{C} \times \mathbb{H}$ $GL_2^+(\mathbb{R}) = \underbrace{\mathbb{C}^*}_{\text{scaling}} \times \underbrace{\mathbb{H}}_{\text{sheering}}$

$g(X) \geq 1$, $\mathrm{Stab}(X) = (\mathrm{Coh}\,X, Z_\mu)\cdot \widetilde{GL_2^+}(\mathbb{R})$
"$\mathrm{Stab}^{\mathrm{Geo}}(X)$" $\simeq \mathbb{C} \times \mathbb{H}$

$X = \mathbb{P}^1$: $D^b\,\mathbb{P}^1 \xrightarrow{\sim} D^b\,\mathrm{Rep}(\underbrace{\bullet \rightrightarrows \bullet}_{K_2})$

$E^\bullet \longmapsto \mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1), E^\bullet)$

$\rightsquigarrow \sigma = (\mathrm{Rep}\,Q, Z')$

$\rightsquigarrow \mathrm{Stab}(X) \simeq \mathbb{C}^2$

$\boxed{\mathrm{Stab}(\mathbb{P}^1) \supsetneq \mathrm{Stab}^{\mathrm{Geo}}(X)}$

ASIDE: So $\mathrm{Stab}\,\mathbb{P}^1$ bigger than $\mathrm{Stab}(C_g)$ $g \geq 1$, ie. $\mathrm{Stab}(-)$ captures something about how $\mathbb{P}^1$ & C_g are different.

Partial explanation!

Th$\underline{{}^{m}}$ [Lie Fu – Chunyi Li – Xiaolei Zhao]

X has finite Albanese morphism $\Rightarrow$ $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X) = \left\{ \begin{array}{l} \sigma \in \mathrm{Stab}(X) \\ \text{s.t. } \mathcal{O}_x \ \sigma\text{-stable} \\ \forall x \in X \end{array} \right\}$

finite Albanese morphism $\Leftrightarrow$ a finite map to an abelian variety

e.g. C_g $g \geq 1$, abelian varieties ...

Converse true in some examples (e.g. $\mathbb{P}^1$, K3 surfaces, ...) but I expect it to be false

shameless plug In [arxiv 2307.00815, §1.4] I survey what's known about geometric stability

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! any rank 2 $\mathrm{Stab}(\mathcal{D})$ is $\mathbb{C}^2$ or $\mathbb{C} \times \mathbb{H}$

i.e. Z factors via rk 2 lattice, e.g. $\dim X = 1$, Λ: numerical Grothendieck group

"**Conj** $\forall \dim X$, $\mathrm{Stab}(X)$ contractible"

[August – Wemyss, Hirano] use to prove $K(\pi, 1)$ conjecture

Q: What is Stab$(D^b(pt))$?

$K(\mathcal{A}) = \mathbb{Z}$ (! $\forall$ ♡ $\mathcal{A}$, $K(\mathcal{A}) = K(\mathcal{D})$)

($D^b(pt)$ is like the category of graded vector spaces)

EXERCISE • $\mathcal{A} = \mathrm{Coh}(pt)[n]$

• $\mathrm{Stab}(pt) \simeq \mathbb{C}$

Q: Given $X \xrightarrow{f} Y$, $\mathcal{A}_Y$: ♡ on $D^b(Y)$

Is $f^*\mathcal{A}_Y$ a heart on $D^b(X)$?

[Bridgeland: Flops & derived cats] also arxiv 1103.2444

↳ "$f^*\mathcal{A}_Y$ not a heart but can modify it to make one"

[Macrì - Mehrotra - Stellari '08]:

$X \xrightarrow{f} Y$ + assumptions

$(P_X, Z_X) = \sigma_X \in \mathrm{Stab}(X)$

$\rightsquigarrow$ define $f^{-1}\sigma_X =: \sigma_Y = (P_Y, Z_Y)$

$$P_Y(\phi) = \{\mathcal{E} \in D^b(Y) : f^*\mathcal{E} \in P_X(\phi)\}$$

also [Polishchuk, constant families of t-structures Theorem 2.1.1]

Q: What is wall crossing? (& stabs $\mathbb{P}^1$ cont.)

Q: finite quiver, $Q_0 = \{0, \dots, n\}$

Consider $\mathrm{Rep}_{\mathbb{C}} Q$

A K_2: $0 \overset{x}{\underset{y}{\rightrightarrows}} 1$

$\underline{V} \in \mathrm{Rep}_{\mathbb{C}} K_2$: $(V_0, V_1, \phi_x, \phi_y : V_0 \to V_1)$

e.g. $\mathbb{C} \overset{\lambda \in \mathbb{C}}{\underset{\mu \in \mathbb{C}}{\rightrightarrows}} \mathbb{C}$ (1,1) $\qquad \mathbb{C}^n \overset{\phi_x}{\underset{\phi_y}{\rightrightarrows}} \mathbb{C}^m$

⚠ $d(\underline{V}) = (\dim V_0, \dim V_1)$

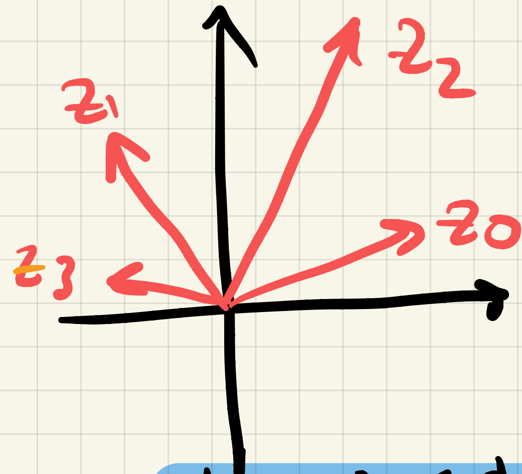
$(1,1)$: $(\lambda, \mu) \neq (0,0)$ – $\mathbb{P}^1_{\mathbb{C}}$ family of reps w/ class $(1,1)$

$\mathbb{C} \overset{0}{\underset{0}{\rightrightarrows}} \mathbb{C} \rightsquigarrow M(1,1)$ is 1-dim variety + 0-dim.

want better behaved moduli $\rightsquigarrow$ use stability conditions

IN GENERAL:

Pick $z_0, \ldots, z_n \in \mathbb{H}$



$Z(\underline{V}) = \sum_{i=0}^{n} \dim V_i \cdot z_i$

$= r e^{i\pi\phi}$

$\underline{V}$ is stable if $\underline{W} \subsetneq \underline{V} \Rightarrow \phi(\underline{W}) < \phi(\underline{V})$

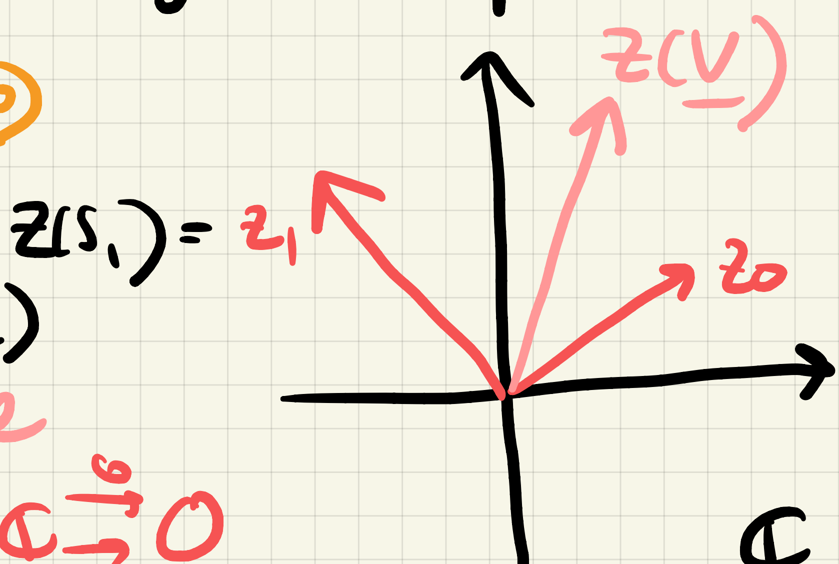
(A): Pick $z_0, z_1 \in \mathbb{H}$

$$S_1 \quad 0 \overset{0}{\underset{0}{\rightrightarrows}} \mathbb{C} \qquad \text{simple rep.}$$

$$\begin{array}{ccc} 0 & \rightrightarrows & \mathbb{C} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \underline{V} \quad \mathbb{C}^n & \overset{d_x}{\underset{d_y}{\rightrightarrows}} & \mathbb{C}^m \end{array}$$

If $m > 0$, S_0 is always a subrep.

Assume $\phi(z_1) > \phi(z_0)$



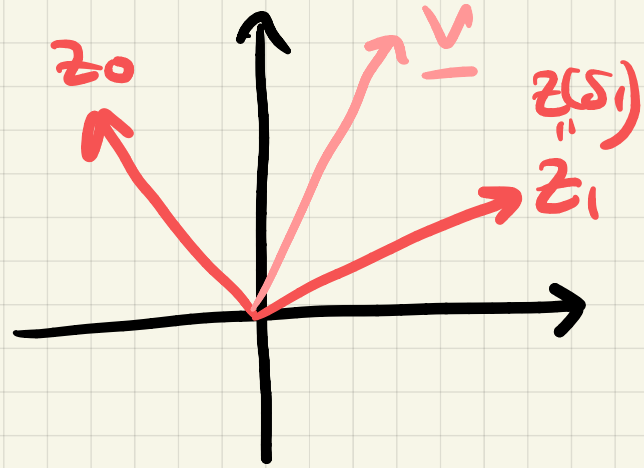
$n > 0 \Rightarrow \phi(S_1) > \phi(\underline{V})$

$\Rightarrow \underline{V}$ not stable

sim. argument for S_0: $\mathbb{C} \overset{0}{\underset{0}{\rightrightarrows}} 0$

$\rightsquigarrow S_0, S_1$ only stable reps.

Assume $\phi(Z_1) < \phi(Z_0)$



consider

$$\underline{S_1} \quad \begin{tikzcd} 0 \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "\alpha"] \\ \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd}$$

$\underline{V}$:

$\phi(\underline{S_1}) < \phi(\underline{V})$ ✓ $(\underline{S_1} \subseteq \underline{V})$

AND

$$\underline{S_0} \quad \begin{tikzcd} \mathbb{C} \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d] & 0 \arrow[d, "0"] \\ \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd}$$

$\underline{V}$

$S_0 \subseteq \underline{V}$

$\iff$ this commutes

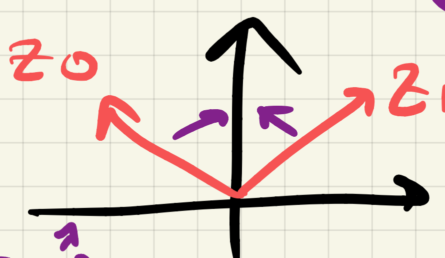
$\iff (\lambda,\mu) = (0,0)$

$\phi(S_0) > \phi(\underline{V})$

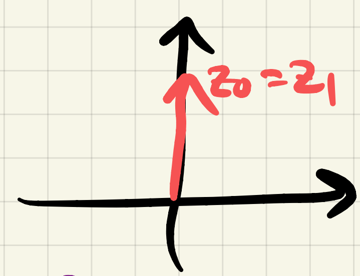
$(\lambda,\mu) \neq (0,0) \Rightarrow \underline{V}$ stable

⚠ These corresp. to $GL_2^+(\mathbb{R}) \cdot \sigma_M$.

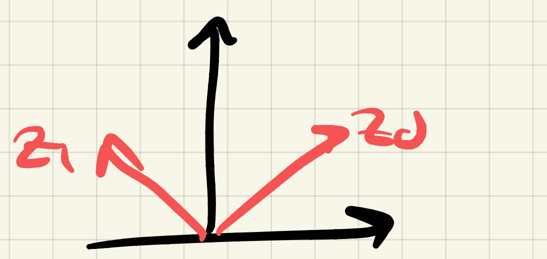
"wall crossing":



$\mathbb{C} \overset{\lambda}{\underset{\mu}{\rightrightarrows}} \mathbb{C}$ stable $(\lambda,\mu) \in \mathbb{P}^1$



$\mathbb{C} \rightrightarrows \mathbb{C}$ semistable (not stable)



$\mathbb{C} \rightrightarrows \mathbb{C}$ unstable (i.e. not semistable)

Q: Refs for quivers & stability?

↳ (:) for quivers, it's called "King stability"

Arend Bayer: "A tour to stability conditions ..."
(on website) ↳ look at exercises

Okada '06 : Stab($\mathbb{P}^1$)

Macrì '07 : some examples of spaces of stability conditions ...

& lots more

Please feel free to email me

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