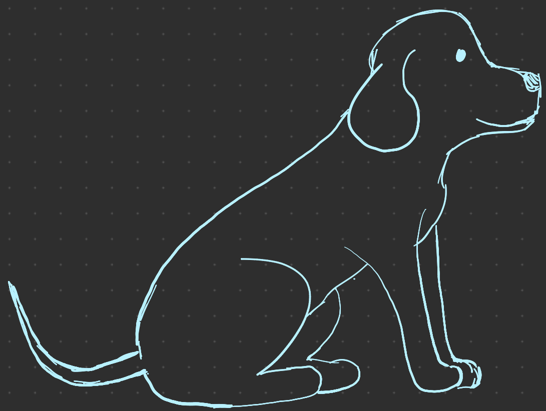
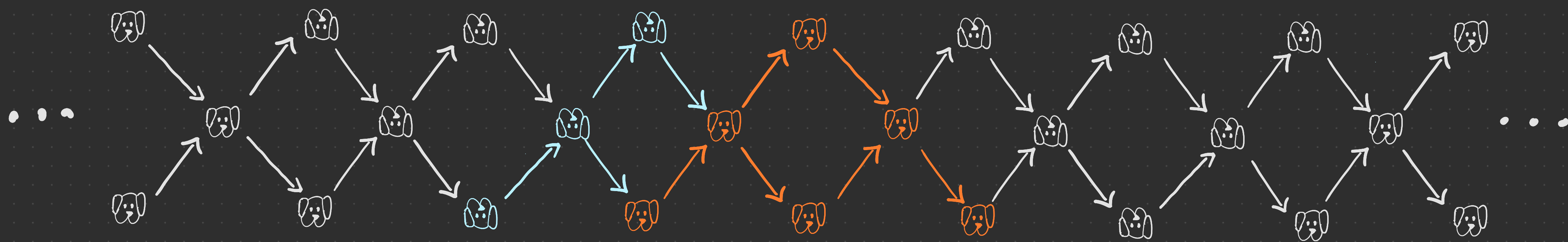
The preprojective algebra of a quiver



Jon Cheah, University of Hong Kong
joncheah@connect.hku.hk

DOGS seminar, 30 Apr 2024



Motivation?

Captures "orientation invariant" rep theory

Cluster algebras on flag strata
[Geiss–Leclerc–Schröer]

$$\Lambda_Q$$

McKay correspondence & Kleinian singularities
[Crawley-Boevey – Holland]

$\#\{\text{ind. } \Lambda\text{-modules}\}$
$= \#\{\text{cluster variables on } \mathbb{C}[N]\}$

Enveloping algebras $\mathcal{U}(\mathfrak{n})$
& semicanonical bases
[Lusztig]

Initial motivation?

$Q = (Q_0, Q_1, s, t)$ a quiver. (Locally finite)

Bernstein, Gelfand, and Ponomarev introduced

- reflection functors $F_i^{\pm} : kQ\text{-mod} \longrightarrow kQ'\text{-mod}$
- Coxeter functors $\Phi^{\pm} = F_{x_n}^{\pm} F_{x_{n-1}}^{\pm} \cdots F_{x_1}^{\pm} : kQ\text{-mod} \longrightarrow kQ\text{-mod}$

so named because they mimic actions of simple reflections and the Coxeter element in the Weyl group.

Initial motivation?

Gelfand and Ponomarev sought an algebra which

- contains kQ as a subalgebra
- as a kQ-module, decomposes as a direct sum of certain* indecomposable kQ-modules.

*those V for which $(\Phi^+)^d(V)$ is projective, for some $d \geq 0$.

These would later come to be known as <u>preprojective modules</u>.

Gelfand & Ponomarev's construction

$\forall a \in Q_1,\ a : s(a) \longrightarrow t(a)$

$a^* \in Q_1^*,\ a^* : t(a) \longrightarrow s(a)$

"opposite arrow"

Doubled quiver $\bar{Q}$, $\bar{Q}_0 = Q_0$

$\bar{Q}_1 = Q_1 \sqcup Q_1^*$

J two-sided ideal of $k\bar{Q}$ gen. by:

$$m_i = \sum_{\substack{a \in Q_1 \\ s(a) = i}} a^* a - \sum_{\substack{b \in Q_1 \\ t(b) = i}} b b^* \quad \forall i \in Q_0$$

If Q_0 finite,

J is principal gen. by

$c = \sum_{i \in Q_0} m_i = \sum_{a \in Q_1} [a^*, a] = \sum_{a \in Q_1} a^* a - a a^*$

Note $\forall i \in Q_0$,

$e_i c e_i = m_i$.

where e_i is idempotent associated to vertex i

$$\boxed{\Lambda = \Lambda_Q := k\bar{Q}/J}$$

Eg: Type A_3

$b: 1 \to 2 \quad a: 2 \to 3$

$ab: 1 \to 2 \to 3$

$Q: 1 \xrightarrow{b} 2 \xrightarrow{a} 3$

$kQ = \langle e_1, e_2, e_3, a, b, ab \rangle$

$$\bar{Q}: \begin{tikzcd} 1 \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "b^*"] \arrow[r, bend left, "a"] & 3 \arrow[l, bend left, "a^*"] \end{tikzcd}$$

$m_1 = b^*b$

$m_2 = a^*a - bb^*$

$m_3 = -aa^*$

$\Lambda = k\bar{Q}/J = \langle e_1, e_2, e_3, a, b, ab, a^*, b^*, a^*a = bb^*, b^*a^* \rangle$

Eg: Type D_4

$$Q: \quad \begin{tikzcd} 1 \arrow[dr, "a"] & & \\ & 4 & 3 \arrow[l, "c"'] \\ 2 \arrow[ur, "b"] & & \end{tikzcd}$$

$$\bar{Q}: \quad \begin{tikzcd} 1 \arrow[dr, bend left, "a"] & & \\ & 4 \arrow[ul, bend left, "a^*"] \arrow[dl, bend left, "b^*"] \arrow[r, bend left, "c^*"] & 3 \arrow[l, bend left, "c"] \\ 2 \arrow[ur, bend left, "b"] & & \end{tikzcd}$$

$$kQ = \langle e_1, e_2, e_3, e_4, a, b, c \rangle$$

$$m_1 = a^*a, \quad m_2 = b^*b, \quad m_3 = c^*c$$

$$m_4 = -\underline{(aa^* + bb^* + cc^*)}$$

$$\Lambda = \left\langle \begin{matrix} \underset{\times 3}{e_i}, \ \underset{\times 3}{x}, & e_4, \ \underset{\times 3}{x^*}, \\ \underset{\times 6}{x^*y}, & aa^*, \ bb^* \\ \underset{\times 3}{xx^*y}, & \underset{\times 3}{x^*yy^*} \\ \underset{\times 3}{y^*xx^*y} & \underset{\times 3}{xx^*yy^*} \end{matrix} \right\rangle$$

$$a^*bb^*c \underset{\text{by } m_4}{=} a^*(\cancel{-aa^*} - \cancel{cc^*})c \underset{\text{by } m_1, m_3}{=} 0$$

Baer-Geigle-Lenzing:

Φ "=" τ AR-translate

"Agree up to a sign change"

Simply define $\Lambda = \bigoplus_{d \geq 0} \mathrm{Hom}_{kQ}(kQ, \tau^{-d}(kQ))$

with algebra structure

$g : kQ \to \tau^{-d'}(kQ)$, $\quad f : kQ \to \tau^{-d}(kQ)$

$$g * f = \tau^{-d}(g) \circ f : kQ \to \tau^{-d}(kQ) \to \tau^{-(d+d')}(kQ)$$

GP $\overset{?}{\longleftrightarrow}$ BGL   Equivalent by [Ringel]

Why does $\Lambda = k\bar{Q}/J$ satisfy the desired properties ?

Assign a grading to $k\bar{Q}$: e_i, a degree 0 for $i \in Q_0$, $a \in Q_1$

[Kleiner] a^* degree 1 for $a \in Q_1$

J is homogeneous, gen. by degree 1, gives a grading on Λ.

$kQ = \Lambda_0$ subalg of deg 0 elements.

Decomposition: $$\boxed{\Lambda = \bigoplus_{d \geq 0} \bigoplus_{i \in Q_0} \Lambda_d e_i = \bigoplus_{d \geq 0} \bigoplus_{i \in Q_0} e_i \Lambda_d}$$

Eg: Type A_3

$Q: 1 \xrightarrow{b} 2 \xrightarrow{a} 3$ $\qquad kQ = \langle e_1, e_2, e_3, a, b, ab \rangle$

$$\bar{Q}: \begin{tikzcd} 1 \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "b^*"] \arrow[r, bend left, "a"] & 3 \arrow[l, bend left, "a^*"] \end{tikzcd}$$

$$\Lambda = \langle \underline{e_1, e_2, e_3, a, b, ab}, a^*, b^*, a^*a = bb^*, b^*a^* \rangle$$

$\Lambda_0 = kQ$

$\Lambda_0 e_1 : k \to k \to k$	$\Lambda_1 e_2 : k \to k \to 0$
$\Lambda_0 e_2 : 0 \to k \to k$	$\Lambda_1 e_3 : 0 \to k \to 0$
$\Lambda_0 e_3 : 0 \to 0 \to k$	$\Lambda_2 e_3 : k \to 0 \to 0$

These are the indecomposable modules of kQ!

Eg: Type A_3

$$Q : 1 \xrightarrow{b} 2 \xrightarrow{a} 3 \qquad kQ = \langle e_1, e_2, e_3, a, b, ab \rangle$$

$$\Lambda = \langle e_1, e_2, e_3, a, b, ab, a^*, b^*, a^*a = bb^*, b^*a^* \rangle$$

AR-quiver of kQ-mod:

$$\begin{tikzcd}
 & & \overset{k\to k\to k}{\Lambda_0 e_1} \arrow[dr] & & \\
 & \overset{0\to k\to k}{\Lambda_0 e_2} \arrow[ur] \arrow[dr] & & \overset{k\to k\to 0}{\Lambda_1 e_2} \arrow[ll, dashed, "\Phi^+"'] \arrow[dr] & \\
\underset{0\to 0\to k}{\Lambda_0 e_3} \arrow[ur] & & \underset{0\to k\to 0}{\Lambda_1 e_3} \arrow[ll, dashed, "\Phi^+"'] \arrow[ur] & & \overset{k\to 0\to 0}{\Lambda_2 e_3} \arrow[ll, dashed, "\Phi^+"']
\end{tikzcd}$$

Coxeter functor takes projective modules to 0

Actually, a derived equiv. which takes projectives out of the heart.

Eg: Type D_4

$$m_1 = a^*a, \quad m_2 = b^*b, \quad m_3 = c^*c$$

$$m_4 = -(aa^* + bb^* + cc^*)$$

$$\overline{Q}: \quad \begin{tikzcd}
1 \arrow[dr, bend left, "a"] & & \\
& 4 \arrow[ul, bend left, "a^*"] \arrow[dl, bend left, "b^*"] \arrow[r, bend left, "c^*"] & 3 \arrow[l, bend left, "c"] \\
2 \arrow[ur, bend left, "b"] & &
\end{tikzcd}$$

$$\Lambda = \left\langle \begin{matrix} e_i, x, & e_4, x^*, \\ x^*y, & aa^*, bb^* \\ xx^*y, & x^*yy^* \\ y^*xx^*y & xx^*yy^* \end{matrix} \right\rangle$$

$$\begin{tikzcd}
& \Lambda_0 e_1 \arrow[ddr] & & \Lambda_1 e_1 \arrow[ll, dashed] \arrow[ddr] & & \Lambda_2 e_1 \arrow[ll, dashed] \\
& \Lambda_0 e_2 \arrow[dr] & & \Lambda_1 e_2 \arrow[ll, dashed] \arrow[dr] & & \Lambda_2 e_2 \arrow[ll, dashed] \\
\Lambda_0 e_4 \arrow[uur] \arrow[ur] \arrow[dr] & & \Lambda_1 e_4 \arrow[ll, dashed] \arrow[uur] \arrow[ur] \arrow[dr] & & \Lambda_2 e_4 \arrow[ll, dashed] \arrow[uur] \arrow[ur] \arrow[dr] & \\
& \Lambda_0 e_3 \arrow[ur] & & \Lambda_1 e_3 \arrow[ll, dashed] \arrow[ur] & & \Lambda_2 e_3 \arrow[ll, dashed]
\end{tikzcd}$$

Eg: Kronecker K_2

$$Q : \begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd} \qquad \bar{Q} : \begin{tikzcd} 1 \arrow[r, bend left=50, "a"] \arrow[r, bend right=50, "b"'] & 2 \arrow[l, bend right=20, "a^*"'] \arrow[l, bend left=20, "b^*"] \end{tikzcd}$$

$\Lambda_0 e_2 : 0 \rightrightarrows k,\quad \Lambda_1 e_2 : k^2 \rightrightarrows k^3,\quad \dots,\quad \Lambda_d e_2 : k^{2d} \rightrightarrows k^{2d+1}$

$\Lambda_0 e_1 : k \rightrightarrows k^2,\quad \Lambda_1 e_1 : k^3 \rightrightarrows k^4,\quad \dots,\quad \Lambda_d e_1 : k^{2d+1} \rightrightarrows k^{2d+2}$

does not include $k \rightrightarrows 0$ injective

Upgrading tools

for $(x_1, \ldots, x_n)$ an "admissible" seq of sources/sinks

Coxeter functor: $\Phi^{\pm} = F^{\pm}_{x_1} \circ \cdots \circ F^{\pm}_{x_n}$

$\downarrow$ Brenner-Butler-Gabriel

Auslander-Reiten translate: $\tau = D\,\underset{\text{Transpose}}{\underline{Tr}}$

$\tau^{-} = Tr\,\underset{\text{v.s. dual}}{\underline{D}}$

$\downarrow$ Happel-Reiten-Van den Bergh

Derived AR-translate: $\tau = S[-1]$

$\tau^{-1} = S^{-1}[1]$

Serre functors

<u>Def</u>: In a Hom-finite k-linear triang. cat. $\mathcal{D}$

a <u>Serre functor</u> is a triangle equivalence $S: \mathcal{D} \to \mathcal{D}$

with bifunctorial $\phi_{X,Y}: \mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(Y, S(X))^*$

$\downarrow \wr$

$\mathrm{Hom}(S^{-1}(Y), X)^*$ for inverse Serre functor

Reiten & Van den Bergh showed that

S exists $\iff$ $\mathcal{D}$ has AR-triangles.

Serre functors on $\mathcal{D}^b(A\text{-mod})$ $(A = kQ)$

If A is a fin. dim. k-alg:

k-v.s. dual

$$S = A^* \overset{L}{\otimes}_A - \ , \qquad S^{-1} = R\mathrm{Hom}_A(A^*, A) \overset{L}{\otimes}_A -$$

If A is homologically smooth,

(i.e. A admits a finite resolution by f.g. projective bimodules)

$$\boxed{S = R\mathrm{Hom}_A(\Theta, -) \ , \qquad S^{-1} = \Theta \overset{L}{\otimes}_A -}$$

where $\Theta = R\mathrm{Hom}_{A^e}(A, A^e)$, $A^e = A \otimes A^{op}$.

Bimodule resolution of $A = kQ$

$A_0 := kQ_0 \simeq k^{|Q_0|}$, $A_1 = k^{|Q_1|}$, A and A_1 are A_0-A_0 bimods.

$$0 \longrightarrow A \otimes_{A_0} A_1 \otimes_{A_0} A \xrightarrow{\phi} A \otimes_{A_0} A \xrightarrow{\mu} A \longrightarrow 0$$

$\ker \mu = \langle w \otimes 1 - 1 \otimes w \rangle_{A\text{-}A}$

$= \langle a \otimes 1 - 1 \otimes a \rangle$

where $\mu : a \otimes b \longmapsto ab$

$\phi : 1 \otimes a \otimes 1 \longmapsto a \otimes 1 - 1 \otimes a$

Bimodule resolution of $A = kQ$

$$0 \longrightarrow A \otimes_{A_0} A_1 \otimes_{A_0} A \xrightarrow{\phi} A \otimes_{A_0} A \xrightarrow{\mu} A \longrightarrow 0$$

$$\phi : 1 \otimes a \otimes 1 \longmapsto a \otimes 1 - 1 \otimes a$$

$$\Theta = R\mathrm{Hom}_{A^e}(A, A^e)$$

Apply $\mathrm{Hom}_{A^e}(-, A^e)$,

$$\begin{tikzcd}
\mathrm{Hom}_{A^e}(A \otimes_{A_0} A, A^e) \arrow[r, "\phi^*"] & \mathrm{Hom}_{A^e}(A \otimes_{A_0} A_1 \otimes_{A_0} A, A^e) \arrow[d, "\cong"] \\
A \otimes_{A_0} A \arrow[u, "\cong"] \arrow[r] & A \otimes_{A_0} \underline{A_1^*} \otimes_{A_0} A
\end{tikzcd}$$

$$\left[\varphi_{a \otimes b} : e_i \otimes e_i \mapsto e_i b \otimes a e_i\right]$$
$$\uparrow$$
$$a \otimes b$$

$$\left[\psi : e_{t(a)} \otimes a \otimes e_{s(a)} \longmapsto e_{t(a)} a'_\psi \otimes a''_\psi e_{s(b)}\right]$$
$$\downarrow$$
$$\sum_a a''_\psi \otimes a^* \otimes a'_\psi$$

$$1 \otimes 1 \longmapsto \sum_{a \in Q_1} 1 \otimes a^* \otimes a - a \otimes a^* \otimes 1$$

dual "flips" A_0 left/right actions

$$e_j \cdot a^* \cdot e_i(x) = a^*(e_i x e_j)$$

Bimodule resolution of $A = kQ$

So $\Theta = RHom_{A^e}(A, A^e)$ is

$$0 \to \overset{0}{A \otimes_{A_0} A} \longrightarrow \overset{1}{A \otimes_{A_0} A_1^* \otimes_{A_0} A} \longrightarrow 0$$

$$1 \otimes 1 \longmapsto \sum_{a \in Q_1} 1 \otimes a^* \otimes a - a \otimes a^* \otimes 1$$

Observe $H^0(\Theta[1]) = \Lambda_1 = \tau^-(kQ)$

$\bigoplus_{d \geq 0} (\Theta[1])^{\otimes d}$ with $\Theta[1]^{\otimes 0} = kQ$

Tensor algebra $\Pi = T_{kQ}(\Theta[1])$. $H^0(\Pi) = \Lambda$.

⚠ $\Pi \simeq_{q.iso} \Lambda_Q$ iff Q is non dynkin

Preprojective property

It is also now clear why τ (or Φ) $(\Lambda_d e_i) = \Lambda_{d-1} e_i$

$$\tau^{-}(\Lambda_d e_i) = \Theta[1] \overset{L}{\otimes}_A \Lambda_d e_i$$

$$= \left[\overset{-1}{A \otimes_{A_0} \Lambda_d e_i} \longrightarrow \overset{0}{A \otimes_{A_0} A_1^* \otimes_{A_0} \Lambda_d e_i} \right]$$

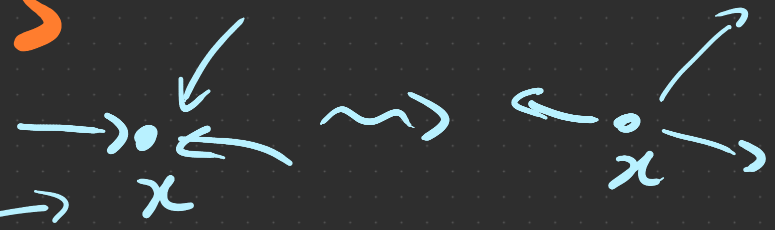
$$1 \otimes x \longmapsto \sum_{a \in Q_1} 1 \otimes a^* \otimes a x - a \otimes a^* \otimes x$$

has cohomology $\Lambda_{d+1} e_i$ in degree 0.

Sketch:

Tilting & Reflections

Reflection functors



For $x \in Q_0$ a sink, have

$$F_x^+ : kQ\text{-mod} \longrightarrow k(\sigma_x Q)\text{-mod}, \quad (V_i, \varphi_a) \longmapsto (\tilde{V}_i, \tilde{\varphi}_a)$$

- $\tilde{V}_i = V_i \ \ \forall\, i \neq x$, $\tilde{\varphi}_{a^*} = \varphi_a$ if $t(a) \neq x$
- $0 \longrightarrow \tilde{V}_x \longrightarrow \bigoplus_{\substack{a \in Q_1 \\ t(a) = x}} V_{s(a)} \xrightarrow{\oplus \varphi_a} V_x \longrightarrow 0$

Dually, $F_x^- : kQ\text{-mod} \to k(\sigma_x X)\text{-mod}$ for $x \in Q_0$ source.

Tilting hearts of t-structures

$\mathcal{A}$ heart of a bounded t-structure on $\mathcal{D}$

Given torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$,

can "tilt" to a new heart $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{ext}$

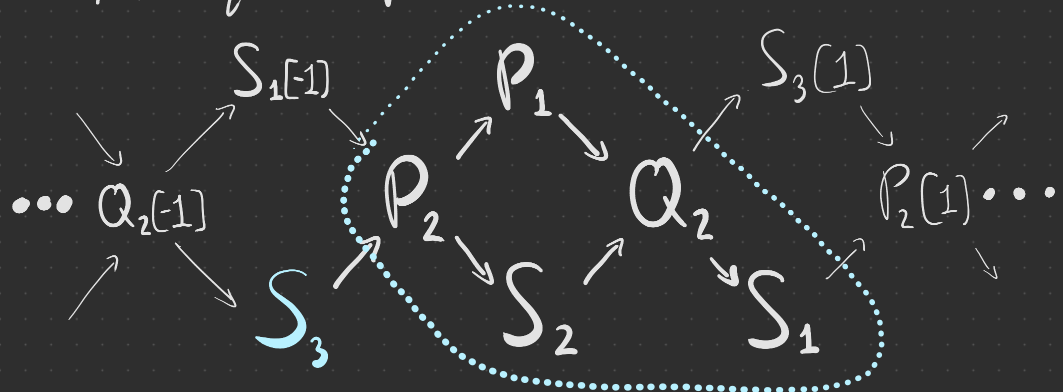
Reflections as tilts

F_x^+ turns out to be tilting wrt

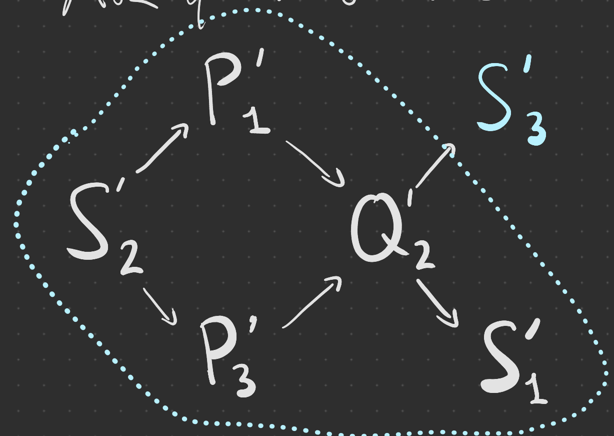
the torsion pair $(\mathcal{T},\mathcal{F}) = \Big(\underset{\{X\in\mathcal{A}\,\mid\,\mathrm{Hom}(X,S_x)=0\}}{\underset{\parallel}{{}^\perp S_x}},\ \operatorname{add} S_x\Big)$

<u>Eg:</u> $Q: 1 \to 2 \to 3$, $x=3$, $\quad Q' = \sigma_x Q: 1 \to 2 \leftarrow 3$

AR-quiver of $\mathcal{D}^b(kQ)$



AR-quiver of kQ'



Compositron of tilts

$\mathcal{A}$ heart of a bdd t-str on a triang. cat $\mathcal{D}$

$\mathcal{A} \longrightarrow \mathcal{B}$ tilt at $(\mathcal{T}, \mathcal{F})$ followed by

$\mathcal{B} \longrightarrow \mathcal{C}$ tilt at $(\mathcal{T}', \mathcal{F}')$ $\boxed{\text{with } \mathcal{F}' \subset \mathcal{T}}$

is given by $\mathcal{A} \longrightarrow \mathcal{C}$ tilt at $(\mathcal{T}'', \mathcal{F}'')$,

where $\mathcal{T}'' = \mathcal{T} \cap \mathcal{T}'$, $\mathcal{F}'' = \{X \in \mathcal{A} \mid t(X) \in \mathcal{F}'\}$

Composition of tilts

Eg:

$Q: 1 \to 2 \to 3$ $\quad \mathcal{A}$

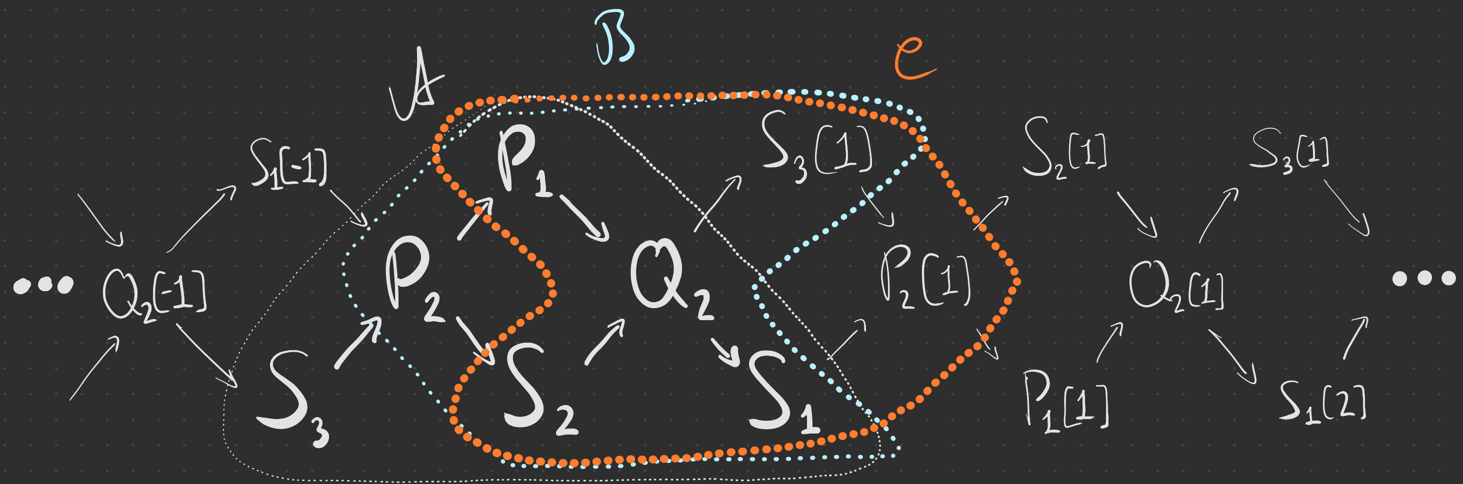
$Q': 1 \to 2 \leftarrow 3$ $\quad \mathcal{B}$

$Q'': 1 \leftarrow 2 \to 3$ $\quad \mathcal{C}$

$\mathcal{A} \to \mathcal{B}$: tilt at $({}^{\perp}S_3, \text{add}\, S_3)$

$\mathcal{B} \to \mathcal{C}$: tilt at $({}^{\perp}S_2, \text{add}\, S_2)$

$\mathcal{A} \to \mathcal{C}$: tilt at $({}^{\perp}\{S_3, P_2\}, \text{add}(P_2 \oplus S_3))$



- The compatibility $\mathcal{F}' \subset \mathcal{T}$ is satisfied for admissible reflection functors.

- View the derived Coxeter functor as a composition of tilts

- This will be a shift of the Serre functor.

Generalisations

- Higher [Iyama–Oppermann] "gl.dim > 1"
- Ginzburg [Ginzburg, Keller–Yang] "different dg-alg, better CY properties"
- Deformed [Crawley-Boevey–Holland] "Deform the relation, related to deformed Kleinian sing."
- Multiplicative [Crawley-Boevey–Shaw, Kaplan–Schedler] "Localise by commutators"